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AN UPPER BOUND FOR THE SUM $\sum_{n=a+1}^{a+H} f(n)$ FOR A CERTAIN CLASS OF FUNCTIONS f

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ABSTRACT. For a certain class of functions $f\colon Z \to C$ an upper bound is obtained for the sum $\sum_{n=a+1}^{a+H} f(n)$. This bound is used to give a proof of a classical inequality due to Pólya and Vinogradov that does not require the value of the modulus of the Gauss sum and to obtain an estimate of the sum of Legendre symbols $\sum_{x=1}^{H}((Rg^x+S)/p)$, where g is a primitive root of the odd prime p, $1 \le H \le p-1$ and RS is not divisible by p.

1. INTRODUCTION

Let $f\colon Z \to C$ be a function satisfying the following three conditions:

(1.1) there exists a positive real number A such that $|f(n)| \le A$ for all $n \in Z$;

(1.2) there exists a positive integer k such that $f(n+k) = f(n)$ for all $n \in Z$;

(1.3) there exists a positive real number B such that

$$\sum_{n=1}^{k}\left|\sum_{r=1}^{h} f(n+r)\right|^2 \le Bhk$$

for all positive integers h.

In §2 we show that such a function f satisfies the following inequality:

Theorem 1. *Let a and H be integers with $1 \le H \le k$. Then, if f satisfies* (1.1)-(1.3), *we have*

$$\left|\sum_{n=a+1}^{a+H} f(n)\right| \le \frac{\sqrt{B}}{2\log 2}\sqrt{k}\log k + 3A\sqrt{k}. \tag{1.4}$$

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In §3 we apply Theorem 1 with $f(n) = \chi(n)$, where χ is a nonprincipal Dirichlet character modulo k, to obtain the following form of the famous inequality first proved independently by Pólya [8] and Vinogradov [10] 70 years ago.

Theorem 2. *Let a and H be integers with $H \geq 1$. If χ is a nonprincipal character modulo k then*

$$\left|\sum_{n=a+1}^{a+H} \chi(n)\right| \leq \frac{\sqrt{k}\log k}{2\log 2} + 3\sqrt{k}. \tag{1.5}$$

What is particularly interesting about our proof of Theorem 2 is that it does not require the value of the modulus of the Gauss sum for a primitive character χ modulo k; namely,

$$\left|\sum_{n=0}^{k-1} \chi(n)\exp(2\pi i n/k)\right| = \sqrt{k}. \tag{1.6}$$

All that is required is for χ to satisfy an inequality of the type (1.3). The result (1.6) is used in the proofs of the Pólya–Vinogradov inequality given in [5, 6, 8, 9, 10] (see also [1, Theorem 13.15, p. 299; 2, Theorem 5.1, p. 320]. No attempt has been made to find the best possible constants multiplying the terms $\sqrt{k}\log k$ and $\sqrt{k}$ in (1.4). It may therefore be possible to improve the constant $1/(2\log 2)$ in (1.5). The form of the Pólya–Vinogradov inequality having the smallest constant multiplying the term $\sqrt{k}\log k$ is due to Hildebrand [5]. Under certain additional assumptions, Burgess [4] has significantly improved the estimate $\sum_{n=a+1}^{a+H}\chi(n) = O(\sqrt{k}\log k)$.

In §4 we use Theorem 1 to estimate the sum of Legendre symbols

$$\sum_{x=1}^{H}\left(\frac{Rg^x + S}{p}\right),$$

where H is an integer with $1 \leq H \leq p-1$, p is an odd prime, g is a primitive root $\pmod p$, and R, S are integers with $RS \not\equiv 0 \pmod p$. We prove

Theorem 3.

$$\left|\sum_{x=1}^{H}\left(\frac{Rg^x + S}{p}\right)\right| \leq \frac{\sqrt{p-1}\log(p-1)}{2\log 2} + 3\sqrt{p-1}.$$

We conclude the introduction by sketching briefly the idea of the proof of Theorem 1. Many values of h are chosen so that relatively many of the inner sums in the inequality (1.3) are partial sums of the sum (1.4) and can be combined to cover most of the range of summation of (1.4) a considerable number of times. An application of the Cauchy–Schwarz inequality then gives inequality (1.4).

2. Proof of Theorem 1

If $1 \leq k \leq 9$ then inequality (1.4) is trivial as

$$\left|\sum_{n=a+1}^{a+H} f(n)\right| \leq \sum_{n=a+1}^{a+H} |f(n)| \leq AH \leq Ak \leq 3A\sqrt{k} \leq \frac{\sqrt{B}}{2\log 2}\sqrt{k}\log k + 3A\sqrt{k}.$$

Thus we may suppose that $k \geq 10$. If $H \leq \sqrt{k}$ then inequality (1.4) is again trivial as

$$\left|\sum_{n=a+1}^{a+H} f(n)\right| \leq AH \leq A\sqrt{k} < 3A\sqrt{k} \leq \frac{k\sqrt{B}}{2\log 2}\sqrt{k}\log k + 3A\sqrt{k}.$$

Thus we may also suppose that $H > \sqrt{k}$. Let

$$S = \{a+1, a+2, \dots, a+H\}, \tag{2.1}$$

and set

$$s = \left[\frac{\log k}{2\log 2}\right], \qquad c = \frac{H}{2^s}, \tag{2.2}$$

$$h_j = 2^{s-j-1}[c], \qquad j = 0, 1, \dots, s-1. \tag{2.3}$$

We note that $s \geq [\log 10/(2\log 2)] = 1$, and that

$$2^s \leq \sqrt{k} < 2^{s+1}, \qquad 1 < c < 2\sqrt{k}. \tag{2.4}$$

From (2.2)–(2.4), we see that the h_j are positive integers satisfying

$$H \geq 2h_0, \qquad H - h_0 - h_1 - \cdots - h_{j-1} \geq 2h_j \qquad (j = 1, \dots, s-1). \tag{2.5}$$

Next we construct $(H - h_0 + 1)s$ sets $S_{i,j}$ $(i = 1, \dots, H - h_0 + 1;\ j = 1, \dots, s)$ such that for $k = 1, \dots, s$,

(2.6)
$S_{i,1} \cup \cdots \cup S_{i,k}$ consists of exactly $h_0 + \cdots + h_{k-1}$ consecutive integers of S.

For $i = 1, \dots, H - h_0 + 1$ we set

$$S_{i,1} = \{a+i, a+i+1, \dots, a+i+h_0-1\}. \tag{2.7}$$

Clearly each $S_{i,1}$ is a sequence of h_0 consecutive integers of S. Now suppose that $S_{i,1}, S_{i,2}, \dots, S_{i,k}$, where k is an integer satisfying $1 \leq k \leq s-1$, have been constructed such that for $j = 1, \dots, k$ the set $S_{i,1} \cup \cdots \cup S_{i,j}$ consists of exactly $h_0 + \cdots + h_{j-1}$ consecutive integers of S. We show how to construct $S_{i,k+1}$ so that $S_{i,1} \cup \cdots \cup S_{i,k+1}$ consists of exactly $h_0 + \cdots + h_k$ consecutive integers of S. Let

$$L = \{s \in S\colon\ s < \min(S_{i,1} \cup \cdots \cup S_{i,k})\},$$
$$R = \{s \in S\colon\ s > \max(S_{i,1} \cup \cdots \cup S_{i,k})\}.$$

Clearly we have

$$\begin{aligned}|L| + |R| &= |S| - |S_{i,1} \cup \cdots \cup S_{i,k}| \\ &= H - (h_0 + h_1 + \cdots + h_{k-1}) \geq 2h_k,\end{aligned}$$

so that $\max(|L|, |R|) \geq h_k$. If $|L| \geq |R|$, so that $|L| \geq h_k$, we set

$$S_{i,k+1} = \{\min(S_{i,1} \cup \cdots \cup S_{i,k}) - h_k, \dots, \min(S_{i,1} \cup \cdots \cup S_{i,k}) - 1\},$$

whereas, if $|L| < |R|$, so that $|R| \geq h_k$, we set

$$S_{i,k+1} = \{\max(S_{i,1} \cup \cdots \cup S_{i,k}) + 1, \dots, \max(S_{i,1} \cup \cdots \cup S_{i,k}) + h_k\}.$$

In both cases $S_{i,k+1}$ consists of h_k consecutive integers of S such that $S_{i,1} \cup \cdots \cup S_{i,k+1}$ comprises $h_0 + \cdots + h_k$ consecutive integers of S. This completes the required construction.

Next, for $i = 1, \ldots, H - h_0 + 1$, we set

$$S_i = S_{i,1} \cup \cdots \cup S_{i,s},$$

and observe that

$$\begin{aligned} |S - S_i| &= \left| S - \bigcup_{j=1}^{s} S_{i,j} \right| \\ &= H - (h_0 + h_1 + \cdots + h_{s-1}) \\ &= c2^s - (2^{s-1} + 2^{s-2} + \cdots + 1)[c] \\ &= c2^s - (2^s - 1)[c] = (c - [c])2^s + [c] \\ &< 2^s + c < \sqrt{k} + 2\sqrt{k}, \quad \text{(by (2.4))} \end{aligned}$$

so that

$$|S - S_i| < 3\sqrt{k}, \qquad i = 1, \ldots, H - h_0 + 1. \tag{2.8}$$

Now, for $i = 1, \ldots, H - h_0 + 1$, we have

$$\left| \sum_{n=a+1}^{a+H} f(n) \right| = \left| \sum_{n \in S} f(n) \right| \le \left| \sum_{n \in S_i} f(n) \right| + \left| \sum_{n \in S - S_i} f(n) \right|,$$

so, appealing to (1.1) and (2.8), we obtain

$$\left| \sum_{n=a+1}^{a+H} f(n) \right| \le \left| \sum_{n \in S_i} f(n) \right| + 3A\sqrt{k} \qquad (i = 1, \ldots, H - h_0 + 1). \tag{2.9}$$

Summing (2.9) over i, we obtain

$$\begin{aligned} (H - h_0 + 1) \left| \sum_{n=a+1}^{a+H} f(n) \right| &= \sum_{i=1}^{H-h_0+1} \left| \sum_{n=a+1}^{a+H} f(n) \right| \\ &\le \sum_{i=1}^{H-h_0+1} \left(\left| \sum_{n \in S_i} f(n) \right| + 3A\sqrt{k} \right) \\ &= \sum_{i=1}^{H-h_0+1} \left| \sum_{n \in S_i} f(n) \right| + 3A\sqrt{k}(H - h_0 + 1), \end{aligned}$$

so that

$$\left| \sum_{n=a+1}^{a+H} f(n) \right| \le \frac{1}{H - h_0 + 1} \sum_{i=1}^{H-h_0+1} \left| \sum_{n \in S_i} f(n) \right| + 3A\sqrt{k}. \tag{2.10}$$

Next, making use of the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\sum_{i=1}^{H-h_0+1}\left|\sum_{n\in S_i} f(n)\right| &\le \sum_{i=1}^{H-h_0+1}\sum_{j=1}^{s}\left|\sum_{n\in S_{i,j}} f(n)\right| \\ &\le \left(\sum_{i=1}^{H-h_0+1}\sum_{j=1}^{s} 1^2\right)^{1/2}\left(\sum_{i=1}^{H-h_0+1}\sum_{j=1}^{s}\left|\sum_{n\in S_{i,j}} f(n)\right|^2\right)^{1/2} \\ &= \sqrt{(H-h_0+1)s}\left(\sum_{i=1}^{H-h_0+1}\sum_{j=1}^{s}\left|\sum_{n\in S_{i,j}} f(n)\right|^2\right)^{1/2},\end{aligned}$$

so that

(2.11)

$$\frac{1}{H-h_0+1}\sum_{i=1}^{H-h_0+1}\left|\sum_{n\in S_i} f(n)\right| \le \frac{\sqrt{s}}{\sqrt{H-h_0+1}}\left(\sum_{i=1}^{H-h_0+1}\sum_{j=1}^{s}\left|\sum_{n\in S_{i,j}} f(n)\right|^2\right)^{1/2}.$$

Furthermore, we have

$$\begin{aligned}\sum_{i=1}^{H-h_0+1}\sum_{j=1}^{s}\left|\sum_{n\in S_{i,j}} f(n)\right|^2 &= \sum_{i=1}^{H-h_0+1}\sum_{j=1}^{s}\sum_{b=a}^{a+H-h_{j-1}}\left|\sum_{\substack{n\in S_{i,j} \\ S_{i,j}=\{b+1,\dots,b+h_{j-1}\}}} f(n)\right|^2 \\ &= \sum_{j=1}^{s}\sum_{b=a}^{a+H-h_{j-1}} N_j(b)\left|\sum_{r=1}^{h_{j-1}} f(b+r)\right|^2,\end{aligned} \tag{2.12}$$

where $N_j(b)$ = number of i $(1 \le i \le H-h_0+1)$ such that $S_{i,j} = \{b+1,\dots,b+h_{j-1}\}$. As $N_j(b) \le 2^{j-1}$, $j = 1,\dots,s$, we have

$$\begin{aligned}&\sum_{j=1}^{s}\sum_{b=a}^{a+H-h_{j-1}} N_j(b)\left|\sum_{r=1}^{h_{j-1}} f(b+r)\right|^2 \\ &\quad\le \sum_{j=1}^{s} 2^{j-1}\sum_{b=a}^{a+H-h_{j-1}}\left|\sum_{r=1}^{h_{j-1}} f(b+r)\right|^2 \\ &\quad\le \sum_{j=1}^{s} 2^{j-1}\sum_{b=a}^{a+k}\left|\sum_{r=1}^{h_{j-1}} f(b+r)\right|^2 \quad (\text{as } H-h_{j-1} < H \le k) \\ &\quad= \sum_{j=1}^{s} 2^{j-1}\sum_{b=1}^{k}\left|\sum_{r=1}^{h_{j-1}} f(b+r)\right|^2 \quad (\text{by } (1.2)) \\ &\quad\le \sum_{j=1}^{s} 2^{j-1} B h_{j-1} k \quad (\text{by } (1.3)),\end{aligned}$$

that is

$$\sum_{j=1}^{s} \sum_{b=a}^{a+H-h_{j-1}} N_j(b) \left| \sum_{r=1}^{h_{j-1}} f(b+r) \right|^2 \leq sBk2^{s-1}[c]. \tag{2.13}$$

Hence, by (2.10)–(2.13), (2.5), (2.3), and (2.2), we have

$$\begin{aligned}
\left| \sum_{n=a+1}^{a+H} f(n) \right| &\leq \frac{\sqrt{s}}{\sqrt{H-h_0+1}} \sqrt{sBk2^{s-1}[c]} + 3A\sqrt{k} \\
&= \frac{s}{\sqrt{H-h_0+1}} \sqrt{B}\sqrt{k}2^{(s-1)/2}\sqrt{[c]} + 3A\sqrt{k} \\
&\leq \frac{s}{\sqrt{2h_1}} \sqrt{B}\sqrt{k}2^{(s-1)/2}\sqrt{[c]} + 3A\sqrt{k} \\
&= \frac{s\sqrt{B}\sqrt{k}2^{(s-1)/2}}{\sqrt{2}\sqrt{h_1}} \frac{\sqrt{h_1}}{2^{(s/2)-1}} + 3A\sqrt{k} \\
&= s\sqrt{B}\sqrt{k} + 3A\sqrt{k} \leq \sqrt{B}\sqrt{k}\frac{\log k}{2\log 2} + 3A\sqrt{k},
\end{aligned}$$

as asserted.

3. Proof of Theorem 2

Let a and H be integers with $H \geq 1$, and let χ be a nonprincipal character modulo k. Now $\sum_n \chi(n) = 0$ if n runs through any complete residue system modulo k, so that $\sum_{n=a+1}^{a+H} \chi(n) = \sum_{n=a+1}^{a+H-k[H/k]} \chi(n)$. Hence we may assume that $H \leq k$. The function $f(n) = \chi(n)$ satisfies (1.1) with $A = 1$, (1.2), and (1.3) with $B = 1$ (see [3]). Theorem 2 now follows immediately from Theorem 1.

4. Proof of Theorem 3

Let p be an odd prime, g a primitive root $\pmod p$, R and S integers with $RS \not\equiv 0 \pmod p$, and H an integer satisfying $1 \leq H \leq p-1$. Next we define

$$f(n) = \left(\frac{Rg^n + S}{p} \right), \tag{4.1}$$

where $(\ /p)$ denotes the Legendre symbol, so that $f\colon Z \to \{-1, 0, 1\}$. Clearly f satisfies (1.1) with $A = 1$ and (1.2) with $k = p-1$. Next we show that f satisfies (1.3) with $B = 1$. First we recall that for $a \not\equiv 0 \pmod p$ we have

$$\sum_{m=0}^{p-1} \left(\frac{am^2 + bm + c}{p} \right) = \begin{cases} -(a/p), & \text{if } b^2 \not\equiv 4ac \pmod p, \\ (p-1)(a/p), & \text{if } b^2 \equiv 4ac \pmod p. \end{cases} \tag{4.2}$$

Then we have

$$\begin{aligned}
\sum_{n=1}^{p-1}\left|\sum_{r=1}^{h} f(n+r)\right|^2 &= \sum_{n=1}^{p-1}\sum_{r=1}^{h}\sum_{s=1}^{h}\left(\frac{Rg^{n+r}+S}{p}\right)\left(\frac{Rg^{n+s}+S}{p}\right)\\
&= \sum_{r,s=1}^{h}\sum_{n=1}^{p-1}\left(\frac{(R^2g^{r+s})g^{2n}+(RS(g^r+g^s))g^n+S^2}{p}\right)\\
&= \sum_{r,s=1}^{h}\sum_{m=1}^{p-1}\left(\frac{(R^2g^{r+s})m^2+(RS(g^r+g^s))m+S^2}{p}\right)\\
&= \sum_{r,s=1}^{h}\left(\sum_{m=0}^{p-1}\left(\frac{(R^2g^{r+s})m^2+(RS(g^r+g^s))m+S^2}{p}\right)-1\right)\\
&= \sum_{r=1}^{h}(p-1)+\sum_{\substack{r,s=1\\ r\neq s}}^{h}(-(-1)^{r+s})-h^2\\
&= h(p-1)-\left(\sum_{r,s=1}^{h}(-1)^{r+s}-\sum_{\substack{r,s=1\\ r=s}}^{h}(-1)^{r+s}\right)-h^2\\
&= h(p-1)-(2[h/2]-h)^2+h-h^2\\
&= hp-h^2-\begin{Bmatrix}0, & h \text{ even}\\ 1, & h \text{ odd}\end{Bmatrix}\\
&\leq h(p-1),
\end{aligned}$$

so that (1.3) holds with $B = 1$. Theorem 3 now follows from Theorem 1.

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