PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 105, Number 4, April 1989

THE REPRESENTATION OF A PAIR OF INTEGERS BY A PAIR OF POSITIVE-DEFINITE BINARY QUADRATIC FORMS

KENNETH HARDY, PIERRE KAPLAN AND KENNETH S. WILLIAMS

(Communicated by William Adams)

ABSTRACT. An explicit formula is given for the number of representations of a pair of positive integers by a representative set of inequivalent pairs of integral positive-definite binary quadratic forms with given invariants.

0. NOTATION

By a *form* we mean a *binary quadratic form* $f = (a, b, c) = aX^2+bXY+cY^2$, which is *integral* (that is a, b, c are integers), *positive definite* (that is $a > 0$, $b^2 - 4ac < 0$) and *primitive* (that is $GCD(a, b, c) = 1$). The discriminant of f, written $\operatorname{disc}(f)$, is the integer $b^2 - 4ac$.

1. INTRODUCTION

Two forms f and f' are said to be equivalent (written $f \sim f'$) if there exists a transformation

$$\tau: \begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} r & s \\ t & u \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}, \tag{1.1}$$

where r, s, t, u are integers satisfying $ru - st = 1$, such that

$$f(rX + sY, tX + uY) = f'(X, Y). \tag{1.2}$$

The transformation τ preserves $\operatorname{disc}(f)$. The relation $\sim$ is an equivalence relation on the set of forms with given discriminant d. It is well known that the number $h(d)$ of equivalence classes is finite. Let

$$f_i = a_iX^2 + b_iXY + c_iY^2, \qquad i = 1, 2, \ldots, h(d), \tag{1.3}$$

be a representative set of inequivalent forms of discriminant d. The positive

Received by the editors July 28, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 10C99.
Research of the first author supported by Natural Sciences and Engineering Research Council of Canada grant A-7823.
Research of the third author supported by Natural Sciences and Engineering Research Council of Canada grant A-7233 and the University of Nancy 1.

©1989 American Mathematical Society
0002-9939/89 $1.00 + $.25 per page

integer m is said to be represented by the form f_i if there exist integers x and y such that

$$m = f_i(x, y). \tag{1.4}$$

The number of pairs (x, y) of integers satisfying (1.4) is denoted by $\psi_d^{(i)}(m)$. Clearly $\psi_d^{(i)}(m)$ is unchanged if the form f_i is replaced by another form equivalent to it. The total number of representations of m by a representative set of inequivalent forms of discriminant d is

$$\psi_d(m) = \sum_{i=1}^{h(d)} \psi_d^{(i)}(m). \tag{1.5}$$

In [1] Dirichlet proved that if $GCD(m, 2d) = 1$ then

$$\psi_d(m) = w(d) \sum_{e|m} \left(\frac{d}{e}\right), \tag{1.6}$$

where e runs through all the positive integers dividing $m, (d/e)$ is the Kronecker symbol and

$$w(d) = \begin{cases} 4, & \text{if } d = -4, \\ 6, & \text{if } d = -3, \\ 2, & \text{if } d \neq -3, -4. \end{cases} \tag{1.7}$$

In this paper we consider the representability of a pair of positive integers (m, M) by pairs of forms and obtain results analogous to Dirichlet's formula (1.6).

2. Pairs of forms

Two pairs of forms $(f, F) = (ax^2 + bxy + cy^2, Ax^2 + Bxy + Cy^2)$ and (f', F') are said to be equivalent, written $(f, F) \sim (f', F')$, if there exists a transformation τ of the type given in (1.1) such that

$$(f(rX+sY, tX+uY), F(rX+sY, tX+uY)) = (f'(X, Y), F'(X, Y)). \tag{2.1}$$

The transformation τ preserves $d = \text{disc}(f) = b^2 - 4ac$, $D = \text{disc}(F) = B^2 - 4AC$, as well as the codiscriminant $\Delta = \text{codisc}(f, F) = bB - 2aC - 2cA$ of the pair (f, F) [3]. From now on we suppose that d, D, and Δ are given and that there are pairs of forms (f, F) with $\text{disc}(f) = d$, $\text{disc}(F) = D$, and $\text{codisc}(f, F) = \Delta$. It is easy to prove [2] that

$$\Delta < 0, \qquad \Delta^2 - dD \geq 0. \tag{2.2}$$

If $\Delta^2 - dD = 0$ it is straightforward [2] to show that $d = D = \Delta$ and that any pair (f, F) with these invariants must have $f = F$. Thus in this case equivalence of pairs of forms reduces to the equivalence of forms described in §1. Thus we may exclude this case and assume from now on that

$$\Delta^2 - dD > 0. \tag{2.3}$$

On the set of pairs of forms (f, F) with specified d, D, and Δ, the relation $\sim$ is an equivalence relation, and the number $h(d, D, \Delta)$ of equivalence classes is finite [3]. A formula for $h(d, D, \Delta)$ has been given by Hardy and Williams [2] in the case when d and D are fundamental discriminants and $GCD(dD, \Delta) = 2^l$ for some $l \geq 0$. We let

$$(2.4)\quad (f_i, F_i) = (a_iX^2 + b_iXY + c_iY^2, A_iX^2 + B_iXY + C_iY^2), \quad i = 1, 2, \ldots, h(d, D, \Delta),$$

be a representative set of inequivalent pairs of forms with given d, D, and Δ. We say that the pair (m, M) of positive integers is represented by the pair (f_i, F_i) if there exist integers x, y such that

$$(2.5)\qquad m = f_i(x, y), \qquad M = F_i(x, y).$$

The number of pairs of integers (x, y) satisfying (2.5) is denoted by $\Psi^{(i)}_{d,D,\Delta}(m, M)$. Clearly $\Psi^{(i)}_{d,D,\Delta}(m, M)$ is unaltered if the pair (f_i, F_i) is replaced by another pair of forms equivalent to (f_i, F_i). The total number of representations of (m, M) by a representative set of inequivalent pairs of forms is

$$(2.6)\qquad \Psi_{d,D,\Delta}(m, M) = \sum_{i=1}^{h(d,D,\Delta)} \Psi^{(i)}_{d,D,\Delta}(m, M).$$

We prove the following theorem which gives the value of $\Psi_{d,D,\Delta}(m, M)$ for all positive integers m, M for which

$$(2.7)\qquad GCD(m, 2d(\Delta^2 - dD)) = GCD(M, 2D(\Delta^2 - dD)) = 1.$$

Theorem. (a) *If* $dM^2 - 2\Delta Mm + Dm^2$ *is not a square then*

$$(2.8)\qquad \Psi_{d,D,\Delta}(m, M) = 0.$$

(b) *If* $dM^2 - 2\Delta Mm + Dm^2 = k^2$ *for some integer* k *and*

$$(2.9)\qquad GCD(m, M) = GCD(m, 2d) = GCD(M, 2D) = 1$$

then

$$(2.10)\qquad \Psi_{d,D,\Delta}(m, M) = \begin{cases} 4, & \text{if } k \neq 0, \\ 2, & \text{if } k = 0. \end{cases}$$

(c) *If* $dM^2 - 2\Delta Mm + Dm^2 = k^2$ *for some integer* k *and*

$$(2.11)\qquad GCD(m, 2d(\Delta^2 - dD)) = GCD(M, 2D(\Delta^2 - dD)) = 1$$

then

(2.12)

$$\Psi_{d,D,\Delta}(m, M) = \begin{cases} 4, & \text{if } k \neq 0 \text{ and } GCD(m, M) = l^2 \text{ for some integer } l, \\ 2, & \text{if } k = 0 \text{ and } GCD(m, M) = l^2 \text{ for some integer } l, \\ 0, & \text{if } GCD(m, M) \neq l^2 \text{ for any integer } l. \end{cases}$$

3. Proof of theorem (a)

If $\Psi_{d,D,\Delta}(m, M) \geq 1$ then there are integers x and y and an integer i ($1 \leq i \leq h(d, D, \Delta)$) such that

$$\begin{cases} m = a_i x^2 + b_i xy + c_i y^2 \\ M = A_i x^2 + B_i xy + C_i y^2, \end{cases} \tag{3.1}$$

and so

$$dM^2 - 2\Delta Mm + Dm^2 = k^2, \tag{3.2}$$

where

$$\pm k = (a_i B_i - b_i A_i)x^2 + 2(a_i C_i - c_i A_i)xy + (b_i C_i - c_i B_i)y^2. \tag{3.3}$$

Hence if $dM^2 - 2\Delta Mm + Dm^2$ is not a square, we must have $\Psi_{d,D,\Delta}(m, M) = 0$.

4. Proof of theorem (b)

Throughout this section we assume that m, M are positive integers satisfying (2.9) and that there exists an integer k such that (3.2) holds. The number of pairs of integers $n \pmod{2m}$ and $N \pmod{2M}$ such that

$$n^2 \equiv d \pmod{4m}, \qquad N^2 \equiv D \pmod{4M}, \tag{4.1}$$

and for which

(4.2) there exist representatives satisfying $Mn - mN = k$,

is denoted by $A(m, M)$. We begin by determining $A(m, M)$.

Lemma 1. $A(m, M) = 1$.

Proof. Clearly, for any solution of (4.1) satisfying (4.2), one has

$$Mn \equiv k \pmod{m}, \qquad mN \equiv -k \pmod{M}. \tag{4.3}$$

Conversely, for any pair of integers (n_0, N_0) for which (4.1) and (4.3) hold, we have

$$Mn_0 - mN_0 \equiv k \pmod{m},$$
$$Mn_0 - mN_0 \equiv k \pmod{M},$$
$$Mn_0 - mN_0 \equiv M^2 n_0^2 + m^2 N_0^2 \equiv dM^2 + Dm^2 \equiv k^2 \equiv k \pmod{2} \text{ (by (3.2))},$$

and so

$$Mn_0 - mN_0 \equiv k \pmod{2mM}.$$

Noting that

$$M(n_0 + 2mr) - (N_0 + 2MR) = (Mn_0 - mN_0) + 2mM(r - R),$$

we see that the classes of $n_0 \pmod{2m}$ and $N_0 \pmod{2M}$ contain representatives n and N satisfying $Mn - mN = k$, that is (4.2) holds. Thus we have

$$A(m, M) = B(d, m, M, k)B(D, M, m, -k), \tag{4.4}$$

where $B(d, m, M, k)$ is the number of solutions $n(\bmod 2m)$ of

$$n^2 \equiv d(\bmod 4m), \qquad Mn \equiv k(\bmod m). \tag{4.5}$$

The congruence $Mn \equiv k(\bmod m)$ has a unique solution $n_0(\bmod m)$. For this solution the congruence $n_0^2 \equiv d(\bmod m)$ is automatically true in view of (3.2). The solutions $\bmod 2m$ of $Mn \equiv k(\bmod m)$ are given by

$$n_0 + \varepsilon m, \qquad \varepsilon = 0 \text{ or } 1.$$

These solutions satisfy $n^2 \equiv d(\bmod 4)$ for the unique value of ε such that

$$(n_0 + \varepsilon)^2 \equiv d(\bmod 4).$$

Thus we have $B(d, m, M, k) = 1$ and similarly $B(D, M, m, -k) = 1$. Hence (4.4) gives $A(m, M) = 1$ as required. □

The next lemma gives the automorphs of a pair of forms (f, F).

Lemma 2. *The only transformations*

$$\tau: \begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} r & s \\ t & u \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \qquad (ru - st = 1)$$

mapping the pair of forms (f, F) *into itself are given by*

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix} = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Proof. If $d \neq -3, -4$ the only automorphs of the form $f = ax^2 + bxy + cy^2$ of discriminant d are

$$\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus the assertion of the lemma is clear unless $(d, D) = (-3, -3)$, $(-3, -4)$, $(-4, -3)$ or $(-4, -4)$.

We just treat the case $(d, D) = (-3, -3)$ as the other cases can be treated similarly. As every form of discriminant -3 is equivalent to the form $(1, 1, 1)$ we may suppose by applying a suitable transformation to f that $f = (1, 1, 1)$. The only automorphs of f are

$$\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \pm \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \qquad \text{and} \qquad \pm \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

The second of these transforms $F = (A, B, C)$ into $(C, -B+2C, A-B+C)$ and so can only be an automorph for the pair (f, F) if $A = C$, $B = -B+2C$, $C = A - B + C$, that is $A = B = C$, i.e., $F = (1, 1, 1)$, and thus $d = D = \Delta = -3$ which is impossible as $\Delta^2 - dD \neq 0$. The third mapping transforms $F = (A, B, C)$ into $(A - B + C, 2A - B, A)$, and, exactly as above, we see that it cannot be an automorph of the pair (f, F). This completes the proof of Lemma 2. □

The next lemma is easily checked.

Lemma 3. *If* $d = n^2 - 4ml$, $D = N^2 - 4ML$ *then the following is an identity*

$$dM^2 + Dm^2 - (mN - Mn)^2 = 2mM(nN - 2mL - 2Ml).$$

We are now ready to prove Theorem (b). If (x, y) is a pair of integers, we set

$$[x, y] = \{(x, y), (-x, -y)\}$$

and for $i = 1, 2, \ldots, h(d, D, \Delta)$ we let

$$S_i = \{[x, y] \mid m = a_i x^2 + b_i xy + c_i y^2, \ M = A_i x^2 + B_i xy + C_i y^2\}.$$

We remark that if $[x, y] \in S_i$ then $GCD(x, y) = 1$ as $GCD(m, M) = 1$. The set of all pairs $([x, y], i)$ with $[x, y] \in S_i$ and $i = 1, 2, \ldots, h(d, D, \Delta)$ is denoted by S. Clearly we have

(4.6) $$\operatorname{card}(S) = \tfrac{1}{2}\Psi_{d,D,\Delta}(m, M).$$

Recalling that m and M are positive integers satisfying (2.9) and for which $dM^2 - 2\Delta Mm + Dm^2 = k^2$ is solvable, we set

$$C_{m,M} = \{(n(\operatorname{mod} 2m), \ N(\operatorname{mod} 2M)) \mid n^2 \equiv d(\operatorname{mod} 4m), \ N^2 \equiv D(\operatorname{mod} 4M), \ Mn - mN = \pm k\}.$$

By Lemma 1 we have

(4.7) $$\operatorname{card}(C_{m,M}) = \begin{cases} 2, & \text{if } k \neq 0, \\ 1, & \text{if } k = 0. \end{cases}$$

Next we define a mapping $T: S \to C_{m,M}$ as follows: if $[x, y] \in S_i$, where $1 \leq i \leq h(d, D, \Delta)$, then

(4.8) $$T(([x, y], i)) = (n(\operatorname{mod} 2m), \ N(\operatorname{mod} 2M)),$$

where

(4.9) $$n = 2a_i x\mu + b_i(x\lambda + y\mu) + 2c_i y\lambda, \qquad N = 2A_i x\mu + B_i(x\lambda + y\mu) + 2C_i y\lambda,$$

and λ, μ are integers such that

(4.10) $$\lambda x - \mu y = 1.$$

We must show that T is well defined and that $\operatorname{range}(T) \subseteq C_{m,M}$. To see that T is well defined we have only to note that if (λ, μ) is replaced by another solution $(\lambda + ty, \mu + tx)$ of (4.10) then n and N are unchanged $(\operatorname{mod} 2)$, and if (x, y) is replaced by $(-x, -y)$ then (λ, μ) can be replaced by $(-\lambda, -\mu)$ and n and N remain the same.

Next we show that T maps into $C_{m,M}$. By the transformation

$$\begin{pmatrix} x & \mu \\ y & \lambda \end{pmatrix}$$

the pair of forms $((a_i, b_i, c_i), (A_i, B_i, C_i))$ becomes the pair $((m, n, l), (M, N, L))$, where

$$l = \frac{n^2 - d}{4m}, \qquad L = \frac{N^2 - D}{4M}, \tag{4.11}$$

and so $n^2 \equiv d \pmod{4m}$, $N^2 \equiv D \pmod{4M}$. As $\Delta = nN - 2mL - 2Ml$, by (3.2) and Lemma 3, we have $Mn - mN = \pm k$.

Now we prove that T maps onto $C_{m,M}$. Let $((n \pmod{2m}), N \pmod{2M})) \in C_{m,M}$ so that $n^2 \equiv d \pmod{4m}$, $N^2 \equiv D \pmod{4M}$, $Mn - mN = \pm k$. We define integers l, L as in (4.11). The forms (m, n, l) and (M, N, L) have discriminants d and D, respectively, and, by Lemma 3 and (3.2), their codiscriminant is Δ. Hence, for a unique integer i $(1 \le i \le h(d, D, \Delta))$, we have

$$((m, n, l), (M, N, L)) \sim ((a_i, b_i, c_i), (A_i, B_i, C_i)).$$

If

$$\begin{pmatrix} x & \mu \\ y & \lambda \end{pmatrix},$$

where $\lambda x - \mu y = 1$, is a transformation mapping $((a_i, b_i, c_i), (A_i, B_i, C_i))$ into $((m, n, l), (M, N, L))$ then $[x, y] \in S_i$, and $T(([x, y], i)) = (n \pmod{2m}, N \pmod{2M}))$. This proves that $\text{range}(T) = C_{m,M}$.

Finally we show that T is one-to-one. Suppose that

$$T([x, y], i) = T([x', y'], i').$$

Then there exist integers n, N, n', N', t, T and two transformations

$$\tau = \begin{pmatrix} x & \mu \\ y & \lambda \end{pmatrix} (x\lambda - \mu y = 1), \qquad \tau' = \begin{pmatrix} x' & \mu' \\ y' & \lambda' \end{pmatrix} (x'\lambda' - y'\mu' = 1)$$

such that

$$n = n' + 2tm, \qquad N = N' + 2TM, \tag{4.12}$$

$$((a_i, b_i, c_i), (A_i, B_i, C_i)) \xrightarrow{\tau} ((m, n, l), (M, N, L)), \tag{4.13}$$

$$((a_{i'}, b_{i'}, c_{i'}), (A_{i'}, B_{i'}, C_{i'})) \xrightarrow{\tau'} ((m, n', l'), (M, N', L')), \tag{4.14}$$

$$Mn - mN = \pm k, \qquad Mn' - mN' = \pm k, \tag{4.15}$$

where l, L are defined as in (4.11), and l', L' are defined similarly. Clearly $Mn - mN = \pm(Mn' - mN')$ and we show that

$$Mn - mN = Mn' - mN'. \tag{4.16}$$

For otherwise $Mn - mN = -(Mn' - mN')$ and appealing to (4.12) we obtain $mM(T - t) = Mn - mN$. As $GCD(m, M) = 1$ we see that $m|n$ and $M|N$, and so by (4.15) we have $mM|k$. Hence from (3.2) we have $m|d$ and $M|d$, contradicting $GCD(m, 2d) = GCD(M, 2D) = 1$. This proves (4.16). From (4.12) and (4.16) we deduce that $t = T$ and so

$$\theta = \begin{pmatrix} 1 & 2t \\ 0 & 1 \end{pmatrix}$$

maps $((m,n,l),(M,N,L)) \to ((m,n',l'),(M,N',L'))$, proving that $i = i'$, and that $\tau'^{-1}\theta\tau$ is an automorphism of the pair $((a_i,b_i,c_i),(A_i,B_i,C_i))$. Hence by Lemma 2 we have

$$\begin{pmatrix} 1 & 2t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x & \mu \\ y & \lambda \end{pmatrix} = \pm \begin{pmatrix} x' & \mu' \\ y' & \lambda' \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

implying $[x',y'] = [x,y]$. This completes the proof that T is one-to-one.

Thus T is a bijection from S to $C_{m,M}$ and so by (4.6) and (4.7) we have

$$\frac{1}{2}\Psi_{d,D,\Delta}(m,M) = \text{card}(S) = \text{card}(C_{m,M}) = \begin{cases} 2, & \text{if } k \neq 0, \\ 1, & \text{if } k = 0, \end{cases}$$

completing the proof of Theorem (b).

5. Proof of theorem (c)

Throughout this section we assume that m, M are positive integers satisfying (3.2) and (2.11).

First we show that if $GCD(m,M) \neq l^2$ for any integer l then $\Psi_{d,D,\Delta}(m,M) = 0$. For suppose $\Psi_{d,D,\Delta}(m,M) \geq 1$. Then there exists i $(1 \leq i \leq h(d,D,\Delta))$ and integers x, y such that

$$(5.1)\qquad \begin{aligned} m &= a_i x^2 + b_i xy + c_i y^2, \\ M &= A_i x^2 + B_i xy + C_i y^2. \end{aligned}$$

Also from (3.3) we have

$$(5.2)\qquad \pm k = (a_i B_i - b_i A_i)x^2 + 2(a_i C_i - c_i A_i)xy + (b_i C_i - c_i B_i)y^2.$$

Solving (5.1) and (5.2) for x^2, xy and y^2, we obtain

$$(5.3)\qquad \begin{aligned} (\Delta^2 - dD)x^2 &= 2(c_i D - C_i \Delta)m + 2(C_i d - c_i \Delta)M \mp 2k(b_i C_i - c_i B_i), \\ (\Delta^2 - dD)xy &= (B_i \Delta - b_i D)m + (b_i \Delta - B_i d)M \pm 2k(a_i C_i - c_i A_i), \\ (\Delta^2 - dD)y^2 &= 2(a_i D - A_i \Delta)m + 2(A_i d - a_i \Delta)M \mp 2k(a_i B_i - b_i A_i). \end{aligned}$$

As $GCD(m,M)$ is not a square, there exists a prime p and a non-negative integer r such that $p^{2r+1} \| GCD(m,M)$. As m and M are odd we have $p \neq 2$. Further from (3.2) we see that $p^{2r+1} | k$ and so from (5.3) we have

$$(5.4)\qquad p^{2r+1} | (\Delta^2 - dD)x^2, \qquad p^{2r+1} | (\Delta^2 - dD)y^2.$$

By (2.11) we have $p \nmid \Delta^2 - dD$ and so $p^{r+1} | x$ and $p^{r+1} | y$. Thus from (5.4) we have $p^{2r+2} | m$ and $p^{2r+2} | M$ contradicting $p^{2r+1} \| GCD(m,M)$.

Finally, if $GCD(m,M) = l^2$, for some integer l, then it is easy to check using (5.1), (5.2), and (5.3) that the mapping $(x,y) \to (x/l, y/l)$ is a bijection from the set of representations of (m,M) by a set of inequivalent pairs of forms with invariants d, D, Δ and the set of representations of $(m/l^2, M/l^2)$ by the same set of pairs of forms. Thus we have, by Theorem (b),

$$\Psi_{d,D,\Delta}(m, l) = \Psi_{d,D,\Delta}(m/l^2, M/l^2) = \begin{cases} 4, & \text{if } k/l^2 \neq 0, \\ 2, & \text{if } k/l^2 = 0, \end{cases}$$

$$= \begin{cases} 4, & \text{if } k \neq 0, \\ 2, & \text{if } k = 0, \end{cases}$$

as required. This completes the proof of Theorem (c). □

6. An example

We take $d = -11$, $D = -11$, $\Delta = -19$ so that $\Delta^2 - dD = 240$. Every pair of forms with these invariants is equivalent to exactly one of the pairs

$$((1, 1, 3), (3, 1, 1)),$$
$$((1, 1, 3), (3, 5, 3)),$$
$$((1, 1, 3), (1, -3, 5)),$$
$$((1, 1, 3), (1, 5, 9)),$$

so $h(-11, -11, -19) = 4$.

If we take $m = 97$ and $M = 31$ (so that $GCD(m, M) = GCD(m, 2d) = GCD(M, 2D) = 1$) we have $dM^2 - 2\Delta Mm + Dm^2 = 196$, so $k = \pm 14$. Thus by Theorem (b) we must have $\Psi_{-11,-11,-19}(97, 31) = 4$. Indeed

$$97 = x^2 + xy + 3y^2, \qquad 31 = x^2 - 3xy + 5y^2, \qquad \text{with } (x, y) = \pm(7, 3),$$
$$97 = x^2 + xy + 3y^2, \qquad 31 = x^2 + 5xy + 9y^2, \qquad \text{with } (x, y) = \pm(10, -3).$$

Finally, we remark that the choice $m = M = 3$ shows that the condition $GCD(m, \Delta^2 - dD) = GCD(M, \Delta^2 - dD) = 1$ is necessary in Theorem (c) as

$$3 = x^2 + xy + 3y^2 = 3x^2 + xy + y^2$$

is solvable with $(x, y) = \pm(1, -1)$.

References

1. P. G. L. Dirichlet, *Vorlesungen über Zahlentheorie*, reprinted 4th edition, Chelsea Publishing Company, New York, 1968, p. 229.
2. K. Hardy and K. S. Williams, *The class number of pairs of positive-definite binary quadratic forms*, Acta Arithmetica (to appear).
3. C. Hooley, *On the diophantine equation* $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$, Arch. Math. **19** (1968), 472–478.

Current address (K. Hardy and K. S. Williams): Department of Mathematics and Statistics, Carleton University, Ottawa, Ontario, Canada K1S 5B6

Current address (P. Kaplan): U.E.R. des Sciences Mathématiques, Université de Nancy 1 B.P. 239, 54506 Vandoeuvre lès Nancy, France