TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 311, Number 1, January 1989

DETERMINATION OF ALL IMAGINARY CYCLIC QUARTIC FIELDS WITH CLASS NUMBER 2

KENNETH HARDY, RICHARD H. HUDSON, DAVID RICHMAN AND KENNETH S. WILLIAMS

ABSTRACT. It is proved that there are exactly 8 imaginary cyclic quartic fields with class number 2.

1. INTRODUCTION

Let K be an imaginary cyclic quartic extension of the rational field Q. K has a unique quadratic subfield which we denote by k. The class number of K (resp. k) is denoted by $h(K)$ (resp. $h(k)$). The relative class number of K over k is the positive integer $h^*(K) = h(K)/h(k)$. The conductor of K is denoted by f. In 1972 Uchida [17] proved that if K is a field with $h^*(K) = 1$ then $f < 50,000$. In 1980 Setzer [14] computed the values of $h^*(K)$ for all fields K having $h^*(K) \equiv 1 \pmod 2$ for which $f < 50,000$. He found that $h^*(K) = 1$ for exactly 7 fields K. Since $h(k) = 1$ for these 7 fields, Setzer's work completed the proof of the following theorem.

Theorem 1 (Uchida-Setzer). *If K is an imaginary cyclic quartic field of class number 1, then K is one of the 7 fields*

$$Q\left(\sqrt{-(5+2\sqrt{5})}\right) \qquad (f=5),$$

$$Q\left(\sqrt{-(13+2\sqrt{13})}\right) \qquad (f=13),$$

$$Q\left(\sqrt{-(2+\sqrt{2})}\right) \qquad (f=16),$$

Received by the editors June 8, 1987 and, in revised form, January 25, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 12A50, 12A30; Secondary 12A35.
Key words and phrases. Imaginary cyclic quartic fields, class number.
Research of the first author was supported by Natural Sciences and Engineering Research Council of Canada Grant A-7823.
Research of the fourth author was supported by Natural Sciences and Engineering Research Council of Canada Grant A-7233.

$$Q\left(\sqrt{-(29+2\sqrt{29})}\right) \qquad (f=29),$$
$$Q\left(\sqrt{-(37+6\sqrt{37})}\right) \qquad (f=37),$$
$$Q\left(\sqrt{-(53+2\sqrt{53})}\right) \qquad (f=53),$$
$$Q\left(\sqrt{-(61+6\sqrt{61})}\right) \qquad (f=61).$$

In this paper we determine all the imaginary cyclic quartic fields of class number 2. We prove the following theorem.

Theorem 2. *If K is an imaginary cyclic quartic field of class number 2, then K is one of the 8 fields*

$$Q\left(\sqrt{-(5+\sqrt{5})}\right) \qquad (f=40),$$
$$Q\left(\sqrt{-3(2+\sqrt{2})}\right) \qquad (f=48),$$
$$Q\left(\sqrt{-13(5+2\sqrt{5})}\right) \qquad (f=65),$$
$$Q\left(\sqrt{-5(13+2\sqrt{13})}\right) \qquad (f=65),$$
$$Q\left(\sqrt{-5(2+\sqrt{2})}\right) \qquad (f=80),$$
$$Q\left(\sqrt{-17(5+2\sqrt{5})}\right) \qquad (f=85),$$
$$Q\left(\sqrt{-(13+3\sqrt{13})}\right) \qquad (f=104),$$
$$Q\left(\sqrt{-7(17+4\sqrt{17})}\right) \qquad (f=119).$$

We now describe how this theorem is proved. In §2 we give a formula for $h^*(K)$. In §3 we determine the form of the conductor f of those imaginary cyclic quartic fields K having $h^*(K) \equiv 2 \pmod 4$. We prove

Proposition 1. *Let K be an imaginary cyclic quartic extension of the rational field Q with conductor f. Then $h^*(K) \equiv 2 \pmod 4$ if and only if*

$$f = 8p, \quad \text{where } p \equiv 5 \pmod 8,$$

or

$$f = 16p, \quad \text{where } p \equiv 3 \text{ or } 5 \pmod 8,$$

or

$$f = pq\,,\, (p/q) = -1\,, \quad \textit{where } p \equiv 3 \pmod 4\,,\, q \equiv 1 \pmod 8\,;$$
$$p \equiv 1 \pmod 8\,,\, q \equiv 5 \pmod 8\,; \textit{ or}$$
$$p \equiv q \equiv 5 \pmod 8\,.$$

[p *and* q *denote distinct odd primes.*]

This result overlaps with the work of Brown and Parry [4, 5] in certain cases.

In §4, by extending the arguments used by Uchida in [17], we prove the following result.

Proposition 2. *Let* K *be an imaginary cyclic quartic extension with conductor* f *such that* $h^*(K) \equiv 2 \pmod 4$. *Suppose that*

$$f \geq A\,, \tag{1.1}$$

where

$$A > 64\,. \tag{1.2}$$

Set

$$B = \frac{1}{2\pi} + \frac{\log\log A}{\pi \log A} + \frac{\gamma}{\pi \log A} + \frac{3}{\pi(\log A - 1/\sqrt{A})} + \frac{3}{2\sqrt{A}\log A}\,, \tag{1.3}$$

$$b = \tfrac{3}{2}(1 + \sqrt{2}) = 3.621320343\ldots\,. \tag{1.4}$$

Let c *be a real number satisfying*

$$0 < c < 1\,. \tag{1.5}$$

Let a *be a real number such that*

$$a > \max\left(\frac{2}{c\log A}\,, b\right)\,. \tag{1.6}$$

Set

$$C = \begin{cases} c + 2(c+1)\dfrac{\log\log A}{\log A} + \dfrac{1}{a(\log A)^2} + \dfrac{U}{\log A}\,, & \textit{if } U > 0\,, \\ c + 2(c+1)\dfrac{\log\log A}{\log A} + \dfrac{1}{a(\log A)^2}\,, & \textit{if } U \leq 0\,, \end{cases} \tag{1.7}$$

where

$$U = \log a + 2c\log B - 2\log(c\sqrt{1-c^2}) + \log\left(\frac{1}{2} + \frac{\log\log A}{\log A} + \frac{\gamma}{\log A} + \frac{1}{(V\sqrt{A}\log A - 1)\log A} + \frac{\log V}{\log A}\right)\,,$$

$$V = \frac{1}{\pi\sqrt{2}} + \frac{\sqrt{2}\log\log A}{\pi\log A} + \frac{3}{2\pi((1/\sqrt{8})\log(A/8) - 1/\sqrt{A})} + \frac{\gamma}{\pi\log A} + \frac{2}{\sqrt{A}\log A}\,,$$

and $\gamma = 0.5772156649\ldots$ *is Euler's constant. Set*

$$(1.8)\quad D = \exp\left(\log a + \frac{1}{a\log A} + 2\log\left(1 + \frac{(9+6\sqrt{2})}{2a}\right) + \frac{4(\log A)(ac\log A + 2)C}{(ac\log A - 2)^2} + \log(2\pi^2)\right).$$

Then we have

$$(1.9)\qquad h^*(K) > \frac{f}{D(\log f)^2}.$$

We remark that from (1.4) and (1.6) we have

$$(1.10)\qquad 1 < b < a.$$

Taking

$$(1.11)\qquad A = 416{,}000,$$

$$(1.12)\qquad c = 0.9285,$$

$$(1.13)\qquad a = 20.55,$$

we obtain

$$(1.14)\qquad B = 0.3103355318,$$

$$(1.15)\qquad C = 1.892633982,$$

$$(1.16)\qquad D = 1242.298845,$$

so that, by Proposition 2, we have

$$(1.17)\qquad h^*(K) > 2.000337977 > 2.$$

This proves the following result.

Proposition 3. *Let* K *be an imaginary cyclic quartic extension of* Q *with relative class number* 2. *Then the conductor* f *of* K *satisfies* $f < 416{,}000$.

A computer program was written to calculate the relative class number $h^*(K)$, from the formula given in (3.13), for all imaginary cyclic quartic fields K having conductor $f < 416{,}000$ of one of the forms given in Proposition 1. A description of the computer program is given in §5. Exactly 10 fields K were found to have $h^*(K) = 2$, namely

$$Q\left(\sqrt{-(5+\sqrt{5})}\right)\qquad (f = 40),$$

$$Q\left(\sqrt{-3(2+\sqrt{2})}\right)\qquad (f = 48),$$

$$Q\left(\sqrt{-13(5+2\sqrt{5})}\right)\qquad (f = 65),$$

$$Q\left(\sqrt{-5(13+2\sqrt{13})}\right)\qquad (f = 65),$$

$$Q\left(\sqrt{-5(2+\sqrt{2})}\right) \qquad (f=80),$$
$$Q\left(\sqrt{-(10+3\sqrt{10})}\right) \qquad (f=80),$$
$$Q\left(\sqrt{-17(5+2\sqrt{5})}\right) \qquad (f=85),$$
$$Q\left(\sqrt{-(85+6\sqrt{85})}\right) \qquad (f=85),$$
$$Q\left(\sqrt{-(13+3\sqrt{13})}\right) \qquad (f=104),$$
$$Q\left(\sqrt{-7(17+4\sqrt{17})}\right) \qquad (f=119).$$

Since

$$h(Q(\sqrt{2})) = h(Q(\sqrt{5})) = h(Q(\sqrt{13})) = h(Q(\sqrt{17})) = 1$$

and

$$h(Q(\sqrt{10})) = h(Q(\sqrt{85})) = 2,$$

Theorem 2 follows from Proposition 3 and Theorem 1.

2. Formula for $h^*(K)$

We denote the multiplicative group of residues coprime with f by G. We have

$$G \cong \mathrm{Gal}(Q(e^{2\pi i/f})/Q). \tag{2.1}$$

Further we denote by H the subgroup of G such that

$$H \cong \mathrm{Gal}(Q(e^{2\pi i/f})/K). \tag{2.2}$$

H is a subgroup of index 4 in G and the factor group G/H is cyclic of order 4. We let α be an element of G such that

$$G/H = \langle \alpha H \rangle, \tag{2.3}$$

and we define a character χ on G by

$$\chi(\alpha) = i, \quad \chi(h) = 1 \quad \text{for all } h \in H. \tag{2.4}$$

There are just 4 characters defined on G which are trivial on H, namely,

$$\chi_0, \chi, \chi^2, \chi^3 \qquad (\chi^4 = \chi_0), \tag{2.5}$$

where χ_0 is the principal character on G, that is,

$$\chi_0(g) = 1 \quad \text{for all } g \in G. \tag{2.6}$$

$\chi, \chi^3 = \overline{\chi}$ are quartic, primitive, odd characters of conductor f. χ^2 is a quadratic, even character $(\mathrm{mod}\, f)$. The primitive character $(\chi^2)'$ induced by

χ^2 is given by

$$(2.7)\qquad (\chi^2)'(n) = (m/n), \qquad (n, m) = 1,$$

where m is the discriminant of the unique quadratic subfield of K.

As is customary we write $\zeta(s)$ for Riemann's zeta function, where $s = \sigma + it$ is a complex variable. The function $\zeta(s)$ is meromorphic, its only pole being a simple pole at $s = 1$ with residue 1. If λ is a character $(\text{mod}\, k)$ the Dirichlet L-function corresponding to λ is denoted as usual by $L(s, \lambda)$. If $\lambda \neq \lambda_0$ (the principal character $(\text{mod}\, k)$), $L(s, \lambda)$ is an entire function. The only zeros of $L(s, \lambda)$, apart from the trivial zeros

$$(2.8)\qquad \begin{cases} s = -2n \ (n = 0, 1, 2, \ldots), & \text{if } \lambda(-1) = 1, \\ s = -(2n+1) \ (n = 0, 1, 2, \ldots), & \text{if } \lambda(-1) = -1, \end{cases}$$

lie in the critical strip $0 < \sigma < 1$. The function $L(s, \lambda_0)$ is a meromorphic function, its only pole being a simple pole at $s = 1$. It is well known that

$$(2.9)\qquad L(s, \lambda_0) = \prod_{p|k} \left(1 - \frac{1}{p^s}\right) \zeta(s) \qquad (\sigma > 1).$$

The class number formula for abelian fields (see for example [7, §3]) applied to the field K gives

$$(2.10)\qquad h^*(K) = \frac{f w(K) L(1, \chi) L(1, \chi^3)}{4\pi^2},$$

where $w(K)$ denotes the number of roots of unity in K, that is,

$$(2.11)\qquad w(K) = \begin{cases} 2, & \text{if } f > 5, \\ 10, & \text{if } f = 5. \end{cases}$$

Since $h^*(K) = 1$, for $f = 5$, we can exclude this case from this point on. Thus we have

$$(2.12)\qquad h^*(K) = \frac{f L(1, \chi) L(1, \chi^3)}{2\pi^2}, \qquad f > 5.$$

3. Proof of Proposition 1

It is shown in [6, Theorem 1] (see also [20]) that the imaginary cyclic quartic field K can be written uniquely in the form

$$(3.1)\qquad K = Q\left(\sqrt{A(D + B\sqrt{D})}\right),$$

where A, B, C, D are integers such that

$$(3.2)\qquad \begin{cases} A & \text{is squarefree, odd and negative}, \\ D = B^2 + C^2 & \text{is squarefree}, B > 0, C > 0, \\ A \text{ and } D & \text{are relatively prime}. \end{cases}$$

It should be noted that the letters A, B, C, D used here have nothing to do with the same letters used in Proposition 2. Moreover it is proved in [7, Theorem 5] that the conductor f of K is given by

(3.3) $$f = 2^l |A| D,$$

where

(3.4) $$l = \begin{cases} 3, & \text{if } D \equiv 2 \pmod 8 \text{ or } D \equiv 1 \pmod 4,\ B \equiv 1 \pmod 2, \\ 2, & \text{if } D \equiv 1 \pmod 4,\ B \equiv 0 \pmod 2,\ A + B \equiv 3 \pmod 4, \\ 0, & \text{if } D \equiv 1 \pmod 4,\ B \equiv 0 \pmod 2,\ A + B \equiv 1 \pmod 4. \end{cases}$$

For each prime p, we let $e_p(f)$ denote the largest integer such that $p^{e_p(f)}$ divides f. It is clear from (3.3) and (3.4) that the following statements hold:

(3.5) if p is an odd prime, $e_p(f) = 0$ or 1;

(3.6) $$e_2(f) = 0, 2, 3 \text{ or } 4;$$

(3.7) if $e_2(f) < 4$ then f has a prime factor congruent to $1 \pmod 4$;

(3.8) if f has only one prime factor then either $f = 16$ or f is a prime $\equiv 5 \pmod 8$;

(3.9) if $f = 8p$ (resp. $4p$) for p an odd prime then $p \equiv 1 \pmod 4$ (resp. $p \equiv 1 \pmod 8$);

(3.10) if $f = pq$ for distinct odd primes p and q, then *either* $p \equiv q \equiv 1 \pmod 4$ with at least one of p and q congruent to $5 \pmod 8$ *or* one of p and q is congruent to $1 \pmod 8$ and the other congruent to $3 \pmod 4$;

(3.11) if f has more than one prime factor then $\phi(f) \equiv 0 \pmod{16}$.

We also set

(3.12) $$C_j = \sum_{\substack{0<n<f/2 \\ \chi(n)=i^j}} 1, \quad D_j = \sum_{\substack{0<n<f/4 \\ \chi(n)=i^j}} 1 \qquad (j = 0, 1, 2, 3).$$

It was shown in [7, Theorem 3] that

(3.13) $$h^*(K) = \rho\{(C_0 - C_2)^2 + (C_1 - C_3)^2\},$$

where

(3.14) $$\rho = \begin{cases} 1/2, & \text{if } f = 5, \\ 1/8, & \text{if } f > 5,\ f \text{ even}, \\ 1/2, & \text{if } f > 5,\ f \text{ odd},\ \chi(2) = 1, \\ 1/18, & \text{if } f > 5,\ f \text{ odd},\ \chi(2) = -1, \\ 1/10, & \text{if } f > 5,\ f \text{ odd},\ \chi(2) = \pm i. \end{cases}$$

It was also observed in [7, equation (6.8)] that

(3.15) $$C_0 + C_2 = C_1 + C_3 = \phi(f)/4.$$

Proposition 1 will be established after a succession of lemmas.

Lemma 1. *If* $f \equiv 0 \pmod 2$ *then for any integer* n *we have*

$$\chi(f/2 - n) = \chi(n). \tag{3.16}$$

Proof. By (3.6) we have $f \equiv 0 \pmod 4$. Thus we have $(1 - f/2)^2 \equiv 1 \pmod f$, so $\chi(1 - f/2) = \pm 1$. Since χ is a primitive character $\pmod f$ we must have

$$\chi(1 - f/2) = -1, \tag{3.17}$$

and, as χ is odd, we deduce

$$\chi(f/2 - 1) = 1. \tag{3.18}$$

The assertion (3.16) follows easily from (3.18).

Lemma 2. *If* $f \equiv 0 \pmod 2$ *then*

$$C_j = 2D_j \qquad (j = 0, 1, 2, 3). \tag{3.19}$$

Proof. As f is even, by Lemma 1, we have $\chi(f/2 - n) = \chi(n)$ for all integers n, from which (3.19) follows.

Lemma 3. *If* $f \equiv 0 \pmod 8$ *then for any integer* n *we have*

$$\chi(f/4 - n) = (-1)^{(n-1)/2}\chi(f/4 - 1)\chi(n). \tag{3.20}$$

Proof. We have (appealing to Lemma 1)

$$\begin{aligned}\left(\chi\left(\frac{f}{4} - 1\right)\right)^2 &= \chi\left(1 - \frac{f}{2}\left(1 - \frac{f}{8}\right)\right)\\ &= -\chi\left(\left(1 - \frac{f}{8}\right)\frac{f}{2} - 1\right) \quad (\text{as } \chi(-1) = -1)\\ &= \begin{cases} +1, & \text{if } f \equiv 8 \pmod{16},\\ -1, & \text{if } f \equiv 0 \pmod{16},\end{cases}\end{aligned}$$

so that

$$\chi\left(\frac{f}{4} - 1\right) = \begin{cases} \pm 1, & \text{if } f \equiv 8 \pmod{16},\\ \pm i, & \text{if } f \equiv 0 \pmod{16}.\end{cases} \tag{3.21}$$

Next we note that

$$\chi\left(\frac{f}{4} - 1\right)\chi\left(\frac{f}{4} + 1\right) = \begin{cases} \chi(f/8 \cdot f/2 - 1), & \text{if } f \equiv 8 \pmod{16},\\ \chi(f/16 \cdot f - 1), & \text{if } f \equiv 0 \pmod{16},\end{cases}$$

so that by Lemma 1 we have

$$\chi\left(\frac{f}{4}-1\right)\chi\left(\frac{f}{4}+1\right)=\begin{cases}+1, & \text{if } f\equiv 8 \pmod{16},\\ -1, & \text{if } f\equiv 0 \pmod{16}.\end{cases} \tag{3.22}$$

Then, from (3.21) and (3.22), we obtain

$$\chi(f/4+1)=\chi(f/4-1). \tag{3.23}$$

If $n\equiv 1 \pmod 4$ then $(f/4-1)n\equiv f/4-n \pmod f$, so that $\chi(f/4-n)=\chi(f/4-1)\chi(n)$. If $n\equiv 3 \pmod 4$ then $-(f/4+1)n\equiv f/4-n \pmod f$, so that by (3.23) we have

$$\chi\left(\frac{f}{4}-n\right)=-\chi\left(\frac{f}{4}+1\right)\chi(n)=-\chi\left(\frac{f}{4}-1\right)\chi(n).$$

This proves (3.20) when n is odd. When n is even (3.20) follows trivially as

$$\chi(f/4-n)=\chi(n)=0.$$

Lemma 4. *If $f=16p$, where p is an odd prime, then*

$$D_0+D_2=D_1+D_3=p-1, \tag{3.24}$$

$$D_0+D_1\equiv\begin{cases}0 \pmod 2, & \text{if } p\equiv 1,7 \pmod 8,\\ 1 \pmod 2, & \text{if } p\equiv 3,5 \pmod 8,\end{cases} \tag{3.25}$$

$$h^*(K)=\begin{cases}0 \pmod 4, & \text{if } p\equiv 1,7 \pmod 8,\\ 2 \pmod 4, & \text{if } p\equiv 3,5 \pmod 8.\end{cases} \tag{3.26}$$

Proof. From Lemma 2 and (3.15) we obtain

$$D_0+D_2=D_1+D_3=\phi(f)/8=\phi(16p)/8=p-1,$$

which is (3.24). Next we have

$$D_0+D_1=\sum_{\substack{0<n<f/8\\ \chi(n)=1 \text{ or } i}}1+\sum_{\substack{0<n<f/8\\ \chi(f/4-n)=1 \text{ or } i}}1,$$

that is,

$$D_0+D_1=\sum_{\substack{0<n<f/8\\ \chi(n)=1 \text{ or } i}}1+\sum_{\substack{0<n<f/8\\ \chi(n)=(-1)^{(n+1)/2}\chi(f/4-1)(1 \text{ or } i)}}1 \tag{3.27}$$

by (3.21) and Lemma 3. Now, for $r, s=0,1,2,3$, set

$$S(r,s)=\sum_{\substack{0<n<f/8\\ n\equiv r \pmod 4\\ \chi(n)=i^s}}1. \tag{3.28}$$

Then, from (3.27) and (3.28), we obtain

$$D_0 + D_1 = \begin{cases} S(1,0)+S(1,1)+S(3,0)+S(3,1) \\ \qquad +S(1,0)+S(1,3)+S(3,1)+S(3,2), & \text{if } \chi(f/4-1)=i, \\ S(1,0)+S(1,1)+S(3,0)+S(3,1) \\ \qquad +S(1,1)+S(1,2)+S(3,0)+S(3,3), & \text{if } \chi(f/4-1)=-i, \end{cases}$$

$$= \begin{cases} S(1,1)+S(1,3)+S(3,0)+S(3,2) \pmod 2, & \text{if } \chi(f/4-1)=i, \\ S(1,0)+S(1,2)+S(3,1)+S(3,3) \pmod 2, & \text{if } \chi(f/4-1)=-i, \end{cases}$$

that is,

(3.29) $$D_0 + D_1 \equiv S(1,0)+S(1,2)+S(3,1)+S(3,3) \pmod 2$$

in both cases, as

$$(S(1,0)+S(1,2)) + (S(1,1)+S(1,3)) + (S(3,0)+S(3,2)) \\ + (S(3,1)+S(3,3)) = \sum_{\substack{0<n<f/8 \\ (n,f)=1}} 1 = \sum_{\substack{0<n<2p \\ (n,2p)=1}} 1 = \phi(2p) = p-1 \equiv 0 \pmod 2.$$

Next, we have

$$\sum_{j=0}^{3} S(3,j) = \sum_{\substack{0<n<f/8 \\ (n,f)=1 \\ n\equiv 3 \pmod 4}} 1 = \sum_{\substack{0<n<2p \\ (n,p)=1 \\ n\equiv 3 \pmod 4}} 1$$

$$= \sum_{\substack{0<n<2p \\ n\equiv 3 \pmod 4}} 1 - \begin{cases} 0, & \text{if } p \equiv 1 \pmod 4, \\ 1, & \text{if } p \equiv 3 \pmod 4 \end{cases}$$

$$= \frac{1}{2}(p-1) - \begin{cases} 0, & \text{if } p \equiv 1 \pmod 4, \\ 1, & \text{if } p \equiv 3 \pmod 4 \end{cases}$$

so that

(3.30) $$S(3,1)+S(3,3) \equiv S(3,0)+S(3,2) \pmod 2.$$

Then (3.29) and (3.30) give

$$D_0 + D_1 \equiv S(1,0)+S(1,2)+S(3,0)+S(3,2) \pmod 2,$$

that is

(3.31) $$D_0 + D_1 \equiv \sum_{\substack{0<n<f/8 \\ \chi^2(n)=1}} 1 \pmod 2.$$

Now, for $f = 16p$, where p is an odd prime, we have from (3.3)

(3.32) $$16p = 2^l |A| D.$$

As A is odd and negative, we have either $A = -1$ or $A = -p$. If $A = -1$ then $D = 2^{4-l}p$, and so as $l = 0, 2$ or 3 and D is squarefree, we must have

$l = 3$ and $D = 2p$. Further, as $D = B^2 + C^2$, we must have $p \equiv 1 \pmod 4$. If on the other hand we have $A = -p$ then $D = 2^{4-l}$, and so, as $l = 0, 2$ or 3 and D is squarefree, we must have $l = 3, D = 2$. Hence we have shown that either

$$K = Q\left(\sqrt{-(2p + B\sqrt{2p})}\right), \tag{3.33}$$

where $p \equiv 1 \pmod 4$ is prime, $2p = B^2 + C^2$, $B > 0$, $C > 0$, or

$$K = Q\left(\sqrt{-p(2+\sqrt{2})}\right), \tag{3.34}$$

where p is an odd prime.

Set

$$p^* = \begin{cases} p, & \text{in case (3.33)}, \\ 1, & \text{in case (3.34)}. \end{cases} \tag{3.35}$$

Then the unique quadratic subfield of K is

$$k = Q\left(\sqrt{2p^*}\right) \tag{3.36}$$

of discriminant $8p^*$. Hence, appealing to [7, §3], we have

$$\chi^2(n) = (8p^*/n), \quad \text{for } n > 0 \text{ and } (n, 8p^*) = 1. \tag{3.37}$$

Thus (3.31) can be written

$$D_0 + D_1 \equiv \sum_{\substack{0<n<2p \\ (n,16p)=1 \\ (8p^*/n)=1}} 1 \equiv \sum_{\substack{0<n<2p \\ (n,2p)=1 \\ (2p^*/n)=1}} 1 \pmod 2. \tag{3.38}$$

Next, by the law of quadratic reciprocity, we have for $0 < n < 2p$ and $(n, 2p) = 1$

$$\begin{aligned} \left(\frac{2p^*}{2p-n}\right) &= \left(\frac{2}{2p-n}\right)\left(\frac{p^*}{2p-n}\right) \\ &= \left(\frac{-1}{p}\right)\left(\frac{-2}{n}\right)\left(\frac{2p-n}{p^*}\right) \\ &= \left(\frac{-1}{p}\right)\left(\frac{-2}{n}\right)\left(\frac{-n}{p^*}\right) \\ &= \left(\frac{-1}{np}\right)\left(\frac{2}{n}\right)\left(\frac{n}{p^*}\right) \\ &= \left(\frac{-1}{np}\right)\left(\frac{2}{n}\right)\left(\frac{p^*}{n}\right), \end{aligned}$$

that is,

$$\left(\frac{2p^*}{2p-n}\right) = \left(\frac{-1}{np}\right)\left(\frac{2p^*}{n}\right), \quad 0 < n < 2p, \ (n, 2p) = 1. \tag{3.39}$$

Hence, using (3.39) in (3.38), we obtain

$$\begin{aligned} D_0+D_1 &\equiv \sum_{\substack{0<n<p\\(n,2p)=1\\(2p^*/n)=1}} 1+\sum_{\substack{0<n<p\\(n,2p)=1\\(2p^*/(2p-n))=1}} 1 \pmod 2\\ &\equiv \sum_{\substack{0<n<p\\(n,2p)=1\\(2p^*/n)=1}} 1+\sum_{\substack{0<n<p\\(n,2p)=1\\(2p^*/n)=(-1/np)}} 1 \pmod 2\\ &\equiv \sum_{\substack{0<n<p\\(n,2p)=1\\(-1/n)=-(-1/p)}} 1 \pmod 2\\ &\equiv \begin{cases} \sum_{\substack{0<n<p\\ n\equiv 3 \pmod 4}} 1 \pmod 2, & \text{if } p\equiv 1 \pmod 4,\\ \sum_{\substack{0<n<p\\ n\equiv 1 \pmod 4}} 1 \pmod 2, & \text{if } p\equiv 3 \pmod 4,\end{cases}\\ &\equiv \begin{cases} \frac14(p-1) \pmod 2, & \text{if } p\equiv 1 \pmod 4,\\ \frac14(p+1) \pmod 2, & \text{if } p\equiv 3 \pmod 4,\end{cases}\\ &\equiv \begin{cases} 0 \pmod 2, & \text{if } p\equiv 1,7 \pmod 8,\\ 1 \pmod 2, & \text{if } p\equiv 3,5 \pmod 8,\end{cases}\end{aligned}$$

completing the proof of (3.25).

Finally, by (3.13), (3.14), (3.19), (3.24) and (3.25), we obtain

$$\begin{aligned} 2h^*(K) &= (D_0-D_2)^2+(D_1-D_3)^2\\ &= (2D_0-(p-1))^2+(2D_1-(p-1))^2\\ &= 4(D_0^2+D_1^2)-4(p-1)(D_0+D_1)+2(p-1)^2\\ &\equiv 4(D_0+D_1) \pmod 8\\ &\equiv \begin{cases} 0 \pmod 8, & \text{if } p\equiv 1,7 \pmod 8,\\ 4 \pmod 8, & \text{if } p\equiv 3,5 \pmod 8\end{cases}\end{aligned}$$

proving (3.26).

Lemma 5. *If $f=8p$, where p is an odd prime (necessarily $p\equiv 1 \pmod 4$ by (3.9)) then we have*

(3.40) $$D_0+D_2=D_1+D_3=\tfrac12(p-1),$$

(3.41) $$D_0+D_1\equiv \tfrac14(p-1) \pmod 2,$$

and

(3.42) $$h^*(K)\equiv \begin{cases} 0 \pmod 4, & \text{if } p\equiv 1 \pmod 8,\\ 2 \pmod 4, & \text{if } p\equiv 5 \pmod 8.\end{cases}$$

Proof. From Lemma 2 and (3.15) we obtain

$$D_0+D_2=D_1+D_3=\frac{\phi(f)}{8}=\frac{\phi(8p)}{8}=\frac{p-1}{2},$$

which proves (3.40).

Further we have

$$D_0 + D_1 = \sum_{\substack{0<n<f/4 \\ \chi(n)=1 \text{ or } i}} 1 = \sum_{\substack{0<n<f/8 \\ \chi(n)=1 \text{ or } i}} 1 + \sum_{\substack{0<n<f/8 \\ \chi(f/4-n)=1 \text{ or } i}} 1,$$

that is,

(3.43) $$D_0 + D_1 = \sum_{\substack{0<n<f/8 \\ \chi(n)=1 \text{ or } i}} 1 + \sum_{\substack{0<n<f/8 \\ \chi(n)=(-1)^{(n-1)/2}\chi(f/4-1)(1 \text{ or } i)}} 1,$$

by (3.20) and (3.21). Then, from (3.43) and (3.28), we obtain

$$D_0 + D_1 = \begin{cases} S(1,0)+S(1,1)+S(3,0)+S(3,1) \\ \quad +S(1,0)+S(1,1)+S(3,2)+S(3,3), & \text{if } \chi(f/4-1)=1, \\ S(1,0)+S(1,1)+S(3,0)+S(3,1) \\ \quad +S(1,2)+S(1,3)+S(3,0)+S(3,1), & \text{if } \chi(f/4-1)=-1 \end{cases}$$
$$\equiv \begin{cases} S(3,0)+S(3,1)+S(3,2)+S(3,3) \pmod 2, & \text{if } \chi(f/4-1)=1, \\ S(1,0)+S(1,1)+S(1,2)+S(1,3) \pmod 2, & \text{if } \chi(f/4-1)=-1, \end{cases}$$

that is,

(3.44) $$D_0 + D_1 \equiv S(1,0)+S(1,1)+S(1,2)+S(1,3) \pmod 2$$

in both cases, as

$$\sum_{r=1,3}\sum_{s=0}^{3} S(r,s) = \sum_{\substack{0<n<f/8 \\ (n,f)=1}} 1 = \sum_{\substack{0<n<p \\ n\equiv 1 \pmod 2}} 1 = \frac{p-1}{2} \equiv 0 \pmod 2.$$

Hence we have

$$D_0 + D_1 \equiv \sum_{s=0}^{3} S(1,s) \pmod 2 \equiv \sum_{\substack{0<n<f/8 \\ n\equiv 1 \pmod 4 \\ (n,f)=1}} 1 \pmod 2 \equiv \sum_{\substack{0<n<p \\ n\equiv 1 \pmod 4}} 1 \pmod 2 \equiv \frac{p-1}{4} \pmod 2,$$

completing the proof of (3.41).

Finally, by (3.13), (3.14), (3.19), (3.40) and (3.41), we have

$$\begin{aligned}2h^*(K) &= (D_0 - D_2)^2 + (D_1 - D_3)^2 \\ &= (2D_0 - \tfrac{1}{2}(p-1))^2 + (2D_1 - \tfrac{1}{2}(p-1))^2 \\ &= 4(D_0^2 + D_1^2) - 2(p-1)(D_0 + D_1) + (p-1)^2/2 \\ &\equiv 4(D_0 + D_1) \pmod 8 \\ &\equiv p - 1 \pmod 8\end{aligned}$$

proving (3.42).

Lemma 6. *If* $f = 4p$, *where* p *is an odd prime* (*necessarily* $p \equiv 1 \pmod 8$ *by* (3.9)) *then we have*

$$D_0 + D_2 = D_1 + D_3 = \tfrac{1}{4}(p-1), \tag{3.45}$$

$$D_0 + D_1 \equiv 0 \pmod 2, \tag{3.46}$$

and

$$h^*(K) \equiv 0 \pmod 4. \tag{3.47}$$

Proof. From Lemma 2 and (3.15) we obtain

$$D_0 + D_2 = D_1 + D_3 = \frac{\phi(f)}{8} = \frac{\phi(4p)}{8} = \frac{1}{4}(p-1),$$

which proves (3.45). Now let χ_4 be the character (mod 4) such that

$$\chi_4(1 + pn) = \chi(1 + pn) \quad \text{for all } n \tag{3.48}$$

and let χ_p be the character $(\bmod p)$ such that

$$\chi_p(1 + 4n) = \chi(1 + 4n) \quad \text{for all } n. \tag{3.49}$$

Clearly we have

$$\chi(n) = \chi_4(n)\chi_p(n) \quad \text{for all } n. \tag{3.50}$$

As χ_4 is a nontrivial character (mod 4) we have

$$\chi_4(n) = (-1)^{(n-1)/2}, \quad \text{if } n \equiv 1 \pmod 2. \tag{3.51}$$

Now, by (3.46), we have

$$\begin{aligned}D_0 + D_1 i &= \frac{1}{2}((D_0 - D_2) + (D_1 - D_3)i) \\ &\quad + \frac{1}{2}((D_0 + D_2) + (D_1 + D_3)i) \\ &= \frac{1}{2}((D_0 - D_2) + (D_1 - D_3)i) + \frac{p-1}{8}(1+i) \\ &\equiv \frac{1}{2}((D_0 - D_2) + (D_1 - D_3)i) \pmod{(1+i)Z[i]} \\ &\equiv \frac{1}{2}\sum_{0<n<p} \chi(n) \pmod{(1+i)Z[i]},\end{aligned}$$

that is, by (3.50) and (3.51),

$$(3.52)\quad D_0 + D_1 \equiv \frac{1}{2} \sum_{\substack{0<n<p \\ n\equiv 1 \pmod 2}} \chi_p(n) - \sum_{\substack{0<n<p \\ n\equiv 3 \pmod 4}} \chi_p(n) \quad (\mathrm{mod}(1+i)Z[i]).$$

Note also that

$$(3.53)\quad (1+\chi_p(-1)) \sum_{\substack{0<n<p \\ n\equiv 1 (\mathrm{mod}\ 2)}} \chi_p(n) = \sum_{0<n<p} \chi_p(n) = 0.$$

Since $\chi_4(-1)\chi_p(-1) = \chi(-1) = -1$, and since $\chi_4(-1) = -1$, we have

$$(3.54)\quad \chi_p(-1) = 1.$$

From (3.53) and (3.54) we deduce that

$$(3.55)\quad \sum_{\substack{0<n<p \\ n\equiv 1 \pmod 2}} \chi_p(n) = 0.$$

Then, from (3.52) and (3.55), we deduce

$$\begin{aligned} D_0 + D_1 &\equiv - \sum_{\substack{0<n<p \\ n\equiv 3 \pmod 4}} \chi_p(n) \quad (\mathrm{mod}(1+i)Z[i]) \\ &\equiv \sum_{\substack{0<n<p \\ n\equiv 3 \pmod 4}} 1 \quad (\mathrm{mod}(1+i)Z[i]) \\ &\equiv \frac{1}{4}(p-1) \quad (\mathrm{mod}(1+i)Z[i]), \end{aligned}$$

that is,

$$(3.56)\quad D_0 + D_1 \equiv 0 \quad (\mathrm{mod}(1+i)Z[i]).$$

The congruence (3.56) and the fact that $(1+i)Z[i] \cap Z = 2Z$ imply that

$$(3.57)\quad D_0 + D_1 \equiv 0 \pmod 2,$$

as required.

Finally by (3.13), (3.14), (3.19), (3.45) and (3.46) we have

$$\begin{aligned} 2h^*(K) &= (D_0 - D_2)^2 + (D_1 - D_3)^2 \\ &= (2D_0 - \tfrac{1}{4}(p-1))^2 + (2D_1 - \tfrac{1}{4}(p-1))^2 \\ &= 4(D_0^2 + D_1^2) - (p-1)(D_0 + D_1) + (p-1)^2/8 \\ &\equiv 4(D_0 + D_1) \pmod 8 \\ &\equiv 0 \pmod 8, \end{aligned}$$

proving (3.47).

Lemma 7. *If $f = pq$, where p and q are distinct odd primes, then we have*

$$C_0 + C_2 \equiv C_1 + C_3 \equiv 0 \pmod 4, \tag{3.58}$$

$$C_0 + C_1 \equiv \begin{cases} 0 \pmod 2, & \textit{if } (p/q) = 1, \\ 1 \pmod 2, & \textit{if } (p/q) = -1, \end{cases} \tag{3.59}$$

and

$$h^*(K) \equiv \begin{cases} 0 \pmod 4, & \textit{if } (p/q) = 1, \\ 2 \pmod 4, & \textit{if } (p/q) = -1. \end{cases} \tag{3.60}$$

Proof. We remark that by (3.10) and the law of quadratic reciprocity we have $(p/q) = (q/p)$.

Appealing to (3.10) and (3.15), we obtain

$$C_0 + C_2 = C_1 + C_3 = (p-1)(q-1)/4 \equiv 0 \pmod 4,$$

completing the proof of (3.58).

We now begin the proof of (3.59). Let χ_p (resp. χ_q) be the unique character $(\bmod\, p)$ (resp. $(\bmod\, q)$) such that

$$\chi_p(1+qn) = \chi(1+qn), \quad \text{for all } n \in Z,$$
$$(\text{resp. } \chi_q(1+pn) = \chi(1+pn), \text{ for all } n \in Z).$$

We have

$$\chi(n) = \chi_p(n)\chi_q(n), \quad \text{for all } n \in Z. \tag{3.61}$$

As

$$\begin{aligned} C_0 + C_1 i &= \frac{1}{2}((C_0 - C_2) + (C_1 - C_3)i) + \frac{1}{2}((C_0 + C_2) + (C_1 + C_3)i) \\ &= \frac{1}{2}\left(\sum_{r=0}^{3} C_r i^r\right) + \frac{1}{8}(p-1)(q-1)(1+i) \\ &= \frac{1}{2}\sum_{0<n<f/2} \chi(n) + \frac{1}{8}(p-1)(q-1)(1+i) \\ &= \frac{1}{2}\sum_{0<n<f/2} \chi_p(n)\chi_q(n) + \frac{1}{8}(p-1)(q-1)(1+i), \end{aligned}$$

we have

$$C_0 + C_1 i \equiv \frac{1}{2}\sum_{0<n<f/2} \chi_p(n)\chi_q(n) \pmod{2(1+i)Z[i]}. \tag{3.62}$$

Since χ is odd, by (3.61), we have

$$\chi_p(-1)\chi_q(-1) = -1. \tag{3.63}$$

Moreover, as $\chi_p(-1) = \pm 1$, $\chi_q(-1) = \pm 1$, one of $\chi_p(-1)$ and $\chi_q(-1)$ is $+1$ and the other is -1. We assume without loss of generality that

$$\chi_p(-1) = -1, \qquad \chi_q(-1) = 1. \tag{3.64}$$

The strategy of the proof of (3.59) is to show that

$$\frac{1}{2}\sum_{0<n<f/2} \chi_p(n)\chi_q^2(n) \equiv \frac{1}{2}\left(1-\left(\frac{p}{q}\right)\right) \pmod{(1+i)Z[i]} \tag{3.65}$$

and

$$\frac{1}{2}\sum_{0<n<f/2} \chi_p(n)\chi_q(n) \equiv \frac{1}{2}\sum_{0<n<f/2} \chi_p(n)\chi_q^2(n) \pmod{(1+i)Z[i]}. \tag{3.66}$$

Then, from (3.62), (3.65) and (3.66), we obtain

$$C_0 + C_1 \equiv C_0 + C_1 i \equiv \frac{1}{2}\left(1-\left(\frac{p}{q}\right)\right) \pmod{(1+i)Z[i]} \tag{3.67}$$

from which (3.59) follows.

First we prove (3.65). We have

$$\frac{1}{2}\sum_{0<n<f/2} \chi_p(n)\chi_q^2(n) = \frac{1}{2}\sum_{\substack{0<n<f/2\\ \chi_q^2(n)=1}} \chi_p(n) - \frac{1}{2}\sum_{\substack{0<n<f/2\\ \chi_q^2(n)=-1}} \chi_p(n),$$

giving

$$\frac{1}{2}\sum_{0<n<f/2} \chi_p(n)\chi_q^2(n) = \frac{1}{2}\sum_{\substack{0<n<f/2\\ (n,q)=1}} \chi_p(n) - \sum_{\substack{0<n<f/2\\ \chi_q^2(n)=-1}} \chi_p(n). \tag{3.68}$$

Since $|\chi_p|\chi_q^2$ is a real character, either half the elements of the set $\{n: 0<n<f,\ (n,f)=1\}$ satisfy $\chi_q^2(n)=-1$ or none of them satisfy $\chi_q^2(n)=-1$. This remark and the fact that $|\chi_p|$ and χ_q^2 are even imply that

$$\sum_{\substack{0<n<f/2\\ \chi_q^2(n)=-1}} |\chi_p(n)| = \frac{1}{2}\sum_{\substack{0<n<f\\ \chi_q^2(n)=-1}} |\chi_p(n)| = \frac{1}{4}\phi(f) \text{ or } 0, \tag{3.69}$$

and so we have

$$\sum_{\substack{0<n<f/2\\ \chi_q^2(n)=-1}} \chi_p(n) \equiv 0 \pmod{(1+i)Z[i]}. \tag{3.70}$$

From (3.68) and (3.70), we obtain

$$\begin{aligned}\frac{1}{2}\sum_{0<n<f/2} \chi_p(n)\chi_q^2(n) &\equiv \frac{1}{2}\sum_{\substack{0<n<f/2\\ (n,q)=1}} \chi_p(n) \pmod{(1+i)Z[i]}\\ &\equiv \frac{1}{2}\sum_{0<n<f/2} \chi_p(n) - \frac{1}{2}\sum_{0<n<p/2} \chi_p(qn) \pmod{(1+i)Z[i]},\end{aligned}$$

that is,

$$\frac{1}{2}\sum_{0<n<f/2} \chi_p(n)\chi_q^2(n) \equiv \frac{1}{2}(1-\chi_p(q))\sum_{0<n<p/2} \chi_p(n) \pmod{(1+i)Z[i]}, \tag{3.71}$$

as

$$\sum_{p/2<n<f/2} \chi_p(n) = 0. \tag{3.72}$$

Since p is prime, $(Z/pZ)^*$ is cyclic, and so there is only one nontrivial quadratic character $(\bmod p)$. Thus, for all integers n, we have

$$\left(\frac{n}{p}\right) = \begin{cases} \chi_p(n), & \text{if } \chi_p \text{ is real}, \\ \chi_p^2(n), & \text{if } \chi_p \text{ is nonreal}. \end{cases} \tag{3.73}$$

Suppose at first that χ_p is real. By (3.73) and assumption (3.64) that $\chi_p(-1) = -1$, we have $(-1/p) = -1$, and so $p \equiv 3 \pmod 4$. Hence we have

$$\sum_{0<n<p/2} \chi_p(n) \equiv \sum_{0<n<p/2} |\chi_p(n)| \equiv \frac{1}{2}(p-1) \equiv 1 \pmod{(1+i)Z[i]}. \tag{3.74}$$

From (3.71) and (3.74) we obtain

$$\frac{1}{2} \sum_{0<n<f/2} \chi_p(n)\chi_q^2(n) \equiv \frac{1}{2}\left(1 - \left(\frac{q}{p}\right)\right) \pmod{(1+i)Z[i]}.$$

This congruence and the fact that $(p/q) = (q/p)$ establish (3.65) when χ_p is real.

Now suppose that χ_p is nonreal. By (3.73) we have

$$(-1/p) = \chi_p^2(-1) = \chi_p((-1)^2) = \chi_p(1) = 1,$$

so that $p \equiv 1 \pmod 4$. Since $\chi_p(-1) = -1$ (by (3.64)) and χ_p is a quartic character, -1 cannot be congruent $(\bmod p)$ to a fourth power. Thus we have $(-1)^{(p-1)/4} \not\equiv \pmod p$, $(-1)^{(p-1)/4} \equiv -1 \pmod p$, and so $p \equiv 5 \pmod 8$. Let E_j $(j = 0, 1, 2, 3)$ denote the number of integers n such that $0 < n < p/2$ and $\chi_p(n) = i^j$. Then we have

$$\begin{aligned}(E_0 + E_2) - (E_1 + E_3) &= \sum_{0<n<p/2} \chi_p^2(n) \\ &= \frac{1}{2} \sum_{0<n<p} \chi_p^2(n) \quad (\text{as } \chi_p^2(-1) = 1) \\ &= 0 \quad (\text{as } \chi_p^2 \text{ is nontrivial})\end{aligned}$$

and so, as $E_0 + E_1 + E_2 + E_3 = \frac{1}{2}(p-1)$, we have

$$E_0 + E_2 = E_1 + E_3 = \tfrac{1}{4}(p-1). \tag{3.75}$$

As $p \equiv 5 \pmod 8$, from (3.75), we deduce

$$E_0 + E_2 \equiv E_1 + E_3 \equiv 1 \pmod 2, \tag{3.76}$$

and so

$$\sum_{0<n<p/2} \chi_p(n) \equiv 1 + i \pmod{2Z[i]}. \tag{3.77}$$

By (3.73) we have

$$\chi_p(q) = \begin{cases} \pm 1, & \text{if } (q/p) = 1, \\ \pm i, & \text{if } (q/p) = -1, \end{cases} \tag{3.78}$$

and so

$$\frac{1}{2}(1 - \chi_p(q))(1+i) \equiv \begin{cases} 0 \pmod{(1+i)Z[i]}, & \text{if } (q/p) = 1, \\ 1 \ (\mathrm{mod}(1+i)Z[i], & \text{if } (q/p) = -1. \end{cases} \tag{3.79}$$

From (3.71), (3.77) and (3.79), we obtain

$$\frac{1}{2} \sum_{0<n<f/2} \chi_p(n)\chi_q^2(n) \equiv \frac{1}{2}\left(1 - \left(\frac{q}{p}\right)\right) \quad (\mathrm{mod}(1+i)Z[i]).$$

This congruence and the fact that $(p/q) = (q/p)$ establish (3.65) when χ_p is nonreal. This completes the proof of (3.65).

Now we prove (3.66). We first observe that

$$\begin{aligned} &\frac{1}{2} \sum_{0<n<f/2} \chi_p(n)(\chi_q^2(n) + \chi_q(n)) \\ &\quad = \frac{1}{2}\sum_{j=0}^{3} \sum_{\substack{0<n<f/2 \\ \chi_q(n)=i^j}} \chi_p(n)(\chi_q^2(n) + \chi_q(n)) \\ &\quad = \sum_{\substack{0<n<f/2 \\ \chi_q(n)=1}} \chi_p(n) + \frac{(-1+i)}{2} \sum_{\substack{0<n<f/2 \\ \chi_q(n)=i}} \chi_p(n) + \frac{(-1-i)}{2} \sum_{\substack{0<n<f/2 \\ \chi_q(n)=-i}} \chi_p(n), \end{aligned}$$

so that

$$\begin{aligned} &\frac{1}{2} \sum_{0<n<f/2} \chi_q(n)(\chi_q^2(n) + \chi_q(n)) \\ &\quad = \sum_{\substack{0<n<f/2 \\ \chi_q(n)=1}} \chi_p(n) + i \sum_{\substack{0<n<f/2 \\ \chi_q(n)=i}} \chi_p(n) - \frac{(1+i)}{2} \sum_{\substack{0<n<f/2 \\ \chi_q^2(n)=-1}} \chi_p(n). \end{aligned} \tag{3.80}$$

We next show that

$$\sum_{\substack{0<n<f/2 \\ \chi_q(n)=1}} \chi_p(n) \equiv 0 \quad (\mathrm{mod}(1+i)Z[i]). \tag{3.81}$$

We have

$$\begin{aligned} \sum_{\substack{0<n<f/2 \\ \chi_q(n)=1}} \chi_p(n) &\equiv \sum_{\substack{0<n<f/2 \\ \chi_q(n)=1}} |\chi_p(n)| \quad (\mathrm{mod}(1+i)Z[i]) \\ &\equiv \frac{1}{2} \sum_{\substack{0<n<f \\ \chi_q(n)=1}} |\chi_p(n)| \quad (\mathrm{mod}(1+i)Z[i]) \end{aligned} \tag{3.82}$$

as $\chi_q(-1) = |\chi_p(-1)| = 1$. Since $|\chi_p|\chi_q$ is a nontrivial quartic or quadratic character, either a half or a quarter of the elements in the set $\{n|0 < n < f, (n, f) = 1\}$ satisfy $\chi_q(n) = 1$, depending on whether or not χ_q is real. This remark and (3.82) imply that

$$(3.83) \quad \sum_{\substack{0<n<f/2 \\ \chi_q(n)=1}} \chi_p(n) \equiv \begin{cases} \frac{1}{4}(p-1)(q-1) \pmod{(1+i)Z[i]}, & \text{if } \chi_q \text{ real}, \\ \frac{1}{8}(p-1)(q-1) \pmod{(1+i)Z[i]}, & \text{if } \chi_q \text{ nonreal}, \end{cases}$$

$$\equiv 0 \pmod{(1+i)Z[i]},$$

proving (3.81). A similar argument shows that

$$(3.84) \quad \sum_{\substack{0<n<f/2 \\ \chi_q(n)=i}} \chi_p(n) \equiv 0 \pmod{(1+i)Z[i]}.$$

Using (3.81) and (3.84) in (3.80), we obtain

(3.85)

$$\frac{1}{2} \sum_{0<n<f/2} \chi_p(n)(\chi_q^2(n) + \chi_q(n)) \equiv -\frac{(1+i)}{2} \sum_{\substack{0<n<f/2 \\ \chi_q^2(n)=-1}} \chi_p(n) \pmod{(1+i)Z[i]}.$$

Our next goal is to show that

$$(3.86) \quad \sum_{\substack{0<n<f/2 \\ \chi_q^2(n)=-1}} \chi_p(n) \in 2Z[i].$$

For all integers j and k, we let E_{jk} denote the number of integers n such that $0 < n < f/2$, $\chi_p(n) = i^j$ and $\chi_q^2(n) = (-1)^k$. Then we have

$$\sum_{\substack{0<n<f/2 \\ \chi_q^2(n)=-1}} \chi_p(n) = E_{01} - E_{21} + iE_{11} - iE_{31},$$

so that

$$(3.87) \quad \sum_{\substack{0<n<f/2 \\ \chi_q^2(n)=-1}} \chi_p(n) \equiv E_{01} + E_{21} + i(E_{11} + E_{31}) \pmod{2Z[i]}.$$

The map $n \to (\chi_p^2(n), \chi_q^2(n))$ is a homomorphism from $(Z/pqZ)^*$ to $\{\pm 1\} \times \{\pm 1\}$. Thus the number of elements which map to a point in the range $\mathfrak{R}$ is

$$(p-1)(q-1)/\text{card}(\mathfrak{R}).$$

Note that

$$\begin{aligned} E_{01} + E_{21} &= \text{card}\{n|0 < n < f/2, \chi_p^2(n) = 1, \chi_q^2(n) = -1\} \\ &= \tfrac{1}{2}\text{card}\{n|0 < n < f, \chi_p^2(n) = 1, \chi_q^2(n) = -1\} \\ &= \begin{cases} 0, & \text{if } \chi_q^2 \text{ is trivial}, \\ (p-1)(q-1)/4, & \text{if } \chi_q^2 \text{ is nontrivial and } \chi_p^2 \text{ is trivial}, \\ (p-1)(q-1)/8, & \text{if } \chi_q^2 \text{ and } \chi_p^2 \text{ are both nontrivial}, \end{cases} \end{aligned}$$

so that

$$E_{01} + E_{21} \equiv 0 \pmod 2. \tag{3.88}$$

Similarly we can show that

$$E_{11} + E_{31} \equiv 0 \pmod 2. \tag{3.89}$$

Assertion (3.86) now follows from (3.87), (3.88) and (3.89). Then, from (3.85) and (3.86), we obtain (3.66). This completes the proof of (3.59).

Finally we prove (3.60). By (3.13) and (3.14) we have (as $f = pq$ is odd)

$$2h^*(K) \equiv (C_0 - C_2)^2 + (C_1 - C_3)^2 \pmod 8.$$

Appealing to (3.58) and (3.59), we obtain

$$\begin{aligned} 2h^*(K) &\equiv 4(C_0^2 + C_1^2) \pmod 8) \\ &\equiv 4(C_0 + C_1) \pmod 8) \\ &\equiv \begin{cases} 0 \pmod 8, & \text{if } (p/q) = 1, \\ 4 \pmod 8, & \text{if } (p/q) = -1, \end{cases} \end{aligned}$$

completing the proof of (3.60).

Lemma 8. *If f has at least three prime factors then*

$$\begin{cases} C_0 + C_2 \equiv C_1 + C_3 \equiv 0 \pmod 4, & \textit{if } f \textit{ odd}, \\ D_0 + D_2 = D_1 + D_3 = \phi(f)/8, & \textit{if } f \textit{ even}, \end{cases} \tag{3.90}$$

$$\begin{cases} C_0 + C_1 \equiv 0 \pmod 2, & \textit{if } f \textit{ odd}, \\ D_0 + D_1 \equiv 0 \pmod 2, & \textit{if } f \textit{ even}, \end{cases} \tag{3.91}$$

and

$$h^*(K) \equiv 0 \pmod 4. \tag{3.92}$$

Proof. We consider two cases according as f is odd or even.

(a) f *odd*. First we note that (3.90) in this case follows at once from (3.11) and (3.15). Next we prove (3.91). Since χ is an odd character there must be a prime factor q of f such that $\chi_q(-1) = -1$. Define $f_1 = f/q$. By (3.5) we have $(f_1, q) = 1$ and so $\chi = \chi_q \chi_{f_1}$. Working modulo $(1+i)Z[i]$ we have

$$\begin{aligned} C_0 + C_1 &\equiv C_0 + C_1 i \\ &\equiv \frac{1}{2}(C_0 - C_2 + (C_1 - C_3)i) + \frac{1}{2}((C_0 + C_2) + (C_1 + C_3)i) \\ &\equiv \frac{1}{2} \sum_{0<n<f/2} \chi(n) + \frac{\phi(f)}{8}(1+i), \end{aligned}$$

that is,

$$C_0 + C_1 \equiv \frac{1}{2} \sum_{0<n<f/2} \chi_q(n)\chi_{f_1}(n) \pmod{(1+i)Z[i]}. \tag{3.93}$$

Since $\chi = \chi_q \chi_{f_1}$ and both χ and χ_q are odd, we have $\chi_{f_1}(-1) = 1$ so that χ_{f_1} is even. Therefore $\chi_q^2 \chi_{f_1}$ is an even character and we have

$$\sum_{0<n<f/2} \chi_q^2(n)\chi_{f_1}(n) = \frac{1}{2} \sum_{0<n<f} \chi_q^2(n)\chi_{f_1}(n) = 0. \tag{3.94}$$

From (3.93) and (3.94), we obtain

$$C_0 + C_1 \equiv \frac{1}{2} \sum_{0<n<f/2} (\chi_q(n) + \chi_q^2(n))\chi_{f_1}(n) \pmod{(1+i)Z[i]}. \tag{3.95}$$

As in the proof of Lemma 7, we have

$$\begin{aligned} &\frac{1}{2} \sum_{0<n<f/2} (\chi_q(n) + \chi_q^2(n))\chi_{f_1}(n) \\ &\quad = \sum_{\substack{0<n<f/2 \\ \chi_q(n)=1}} \chi_{f_1}(n) + i \sum_{\substack{0<n<f/2 \\ \chi_q(n)=i}} \chi_{f_1}(n) - \frac{1}{2}(1+i) \sum_{\substack{0<n<f/2 \\ \chi_q^2(n)=-1}} \chi_{f_1}(n) \end{aligned} \tag{3.96}$$

and

$$\sum_{\substack{0<n<f/2 \\ \chi_q^2(n)=-1}} \chi_{f_1}(n) \equiv 0 \pmod{2Z[i]}, \tag{3.97}$$

so that

$$C_0 + C_1 \equiv \sum_{\substack{0<n<f/2 \\ \chi_q(n)=1 \text{ or } i}} \chi_{f_1}(n) \pmod{(1+i)Z[i]}. \tag{3.98}$$

Since $\chi_{f_1}(n)$ is a fourth root of unity whenever $(n, f) = 1$,

$$\chi_{f_1}(n) \equiv 1 \pmod{(1+i)Z[i]},$$

whenever $(n, f) = 1$. Hence we have

$$C_0 + C_1 \equiv G_0 + G_1 \pmod{(1+i)Z[i]}, \tag{3.99}$$

where G_j denotes the number of integers n such that $0 < n < f/2$, $(n, f) = 1$ and $\chi_q(n) = i^j$.

We want to show that

$$G_0 \equiv G_1 \equiv G_2 \equiv G_3 \pmod{2}. \tag{3.100}$$

First we observe that

$$G_0 - G_2 + i(G_1 - G_3) = \sum_{\substack{0<n<f/2 \\ (n,f)=1}} \chi_q(n). \tag{3.101}$$

Let $q = q_1, q_2, \ldots, q_t$ denote the prime factors of f. By assumption we know that $t \geq 3$. By the inclusion-exclusion principle we have

$$\sum_{\substack{0<n<f/2 \\ (n,f)=1}} \chi_q(n) = (1 - \chi_q(q_2))(1 - \chi_q(q_3)) \cdots (1 - \chi_q(q_t)) \sum_{0<n<q/2} \chi_q(n). \tag{3.102}$$

Since $\chi_q(q_2), \ldots, \chi_q(q_t)$ are fourth roots of 1 and since $t \geq 3$, we have $(1-\chi_q(q_2))\cdots(1-\chi_q(q_t)) \in 2Z[i]$. Furthermore, we have

$$(3.103) \qquad \sum_{0<n<q/2} \chi_q(n) \equiv \frac{1}{2}(q-1) \pmod{(1+i)Z[i]}.$$

Hence, when $q \equiv 1 \pmod 4$, we have

$$(3.104) \qquad (1-\chi_q(q_2))\cdots(1-\chi_q(q_t)) \sum_{0<n<q/2} \chi_q(n) \equiv 0 \pmod{2(1+i)Z[i]}.$$

When $q \equiv 3 \pmod 4$, χ_q must be a quadratic character since every quadratic residue is congruent to a fourth power $\bmod q$. Therefore, when $q \equiv 3 \pmod 4$, we have

$$1-\chi_q(q_j) \equiv 0 \pmod 2, \qquad j = 2, \ldots, t,$$

so

$$(1-\chi_q(q_2))\cdots(1-\chi_q(q_t)) \equiv 0 \pmod 4,$$

and thus in this case as well (3.104) holds. From (3.101), (3.102) and (3.104), we obtain

$$(3.105) \qquad G_0 - G_2 \equiv G_1 - G_3 \equiv 0 \pmod 2,$$

$$(3.106) \qquad \tfrac{1}{2}(G_0 - G_2) \equiv \tfrac{1}{2}(G_1 - G_3) \pmod 2.$$

From (3.105) and (3.106) we see that

$$(3.107) \qquad G_0 - G_2 \equiv G_1 - G_3 \equiv 0 \text{ or } 2 \pmod 4.$$

Since $\chi_q^2(-1) = 1$ we have

$$(3.108) \qquad \sum_{\substack{0<n<f/2 \\ (n,f)=1}} \chi_q^2(n) = \frac{1}{2} \sum_{\substack{0<n<f \\ (n,f)=1}} \chi_q^2(n) = 0 \text{ or } \frac{\phi(f)}{2}.$$

Furthermore, we have

$$(3.109) \qquad \sum_{\substack{0<n<f/2 \\ (n,f)=1}} \chi_q^2(n) = (G_0 + G_2) - (G_1 + G_3).$$

From (3.108) and (3.109) we see that either

$$(3.110)(\text{i}) \qquad G_0 + G_2 = G_1 + G_3 = \phi(f)/4$$

or

$$(3.110)(\text{ii}) \qquad G_0 + G_2 = \phi(f)/2, \qquad G_1 + G_3 = 0.$$

As $\phi(f) \equiv 0 \pmod{16}$ we deduce from (3.110) that

$$(3.111) \qquad G_0 + G_2 \equiv G_1 + G_3 \equiv 0 \pmod 4.$$

Adding (3.107) and (3.111), we obtain

$$(3.112) \qquad G_0 \equiv G_1 \equiv (G_0 - G_2)/2 \pmod 2.$$

Congruences (3.107) and (3.112) imply that

(3.113) $$G_0 \equiv G_1 \equiv G_2 \equiv G_3 \pmod 2,$$

and, from (3.99) and (3.113), we have

$$C_0 + C_1 \equiv 0 \pmod{(1+i)Z[i]}.$$

Since $(1+i)Z[i] \cap Z = 2Z$, $C_0 + C_1$ is even, completing the proof of (3.91) in this case.

Finally we prove (3.92) in this case. By (3.13), (3.14), (3.90) and (3.91), we have

$$\begin{aligned} 2h^*(K) &\equiv (C_0 - C_2)^2 + (C_1 - C_3)^2 \pmod 8 \\ &\equiv 4(C_0^2 + C_1^2) \pmod 8 \\ &\equiv 4(C_0 + C_1) \pmod 8 \\ &\equiv 0 \pmod 8, \end{aligned}$$

completing the proof of (3.92) for f odd.

(b) *f even.* First we prove (3.90) in this case. By (3.11), (3.15) and Lemma 2 we have

$$D_0 + D_2 = D_1 + D_3 = \phi(f)/8 \equiv 0 \pmod 2.$$

Next we prove (3.91). In order to do this we let $t_j(F)$ denote the number of elements in the set

$$\{n\colon 1 \le n < F,\ (n, F) = 1,\ n \equiv j \pmod 4\},$$

where F denotes an odd number greater than 1. We begin by proving

(3.114) $$D_0 + D_1 \equiv \begin{cases} t_3(F) \pmod 2, & \text{if } f \equiv 4 \pmod 8, \\ t_{-F}(F) \pmod 2, & \text{if } f \equiv 0 \pmod 8, \end{cases}$$

where F is the largest odd factor of f. Note that by our assumptions on f we have $F > 1$. We begin with the case $f = 4F$. We have $\chi = \chi_4 \chi_F$, where χ_4 and χ_F are the characters (mod 4) and (mod F) respectively defined by

(3.115) $$\begin{cases} \chi_4(1 + Fn) = \chi(1 + Fn), & \text{for all } n, \\ \chi_F(1 + 4n) = \chi(1 + 4n), & \text{for all } n. \end{cases}$$

As χ_4 is a nontrivial character (mod 4) we have

(3.116) $$\chi_4(n) = (-1)^{(n-1)/2} \quad \text{when } n \equiv 1 \pmod 2.$$

Note that

(3.117) $$\begin{aligned} D_0 - D_2 + iD_1 - iD_3 &= \sum_{0<n<F} \chi(n) \\ &= \sum_{\substack{0<n<F \\ n \equiv 1 \pmod 2}} \chi_F(n) - 2 \sum_{\substack{0<n<F \\ n \equiv 3 \pmod 4}} \chi_F(n) \end{aligned}$$

and that

(3.118) $$(1 + \chi_F(-1)) \sum_{\substack{0<n<F \\ n \text{ odd}}} \chi_F(n) = \sum_{0<n<F} \chi_F(n) = 0.$$

Since

$$\chi_F(-1) = \chi(-1)/\chi_4(-1) = -1/-1 = 1\,, \tag{3.119}$$

we see, from (3.118) and (3.119), that

$$\sum_{\substack{0<n<F \\ n \text{ odd}}} \chi_F(n) = 0\,, \tag{3.120}$$

and so, from (3.117) and (3.120), we have

$$\frac{1}{2}(D_0 - D_2 + i(D_1 - D_3)) = -\sum_{\substack{0<n<F \\ n\equiv 3 \pmod 4}} \chi_F(n)\,. \tag{3.121}$$

Further, as

$$\frac{1}{2}(D_0 + D_2 + i(D_1 + D_3)) = \frac{\phi(f)}{16}(1+i) \equiv 0 \pmod{(1+i)Z[i]}\,, \tag{3.122}$$

we obtain by adding (3.121) and (3.122)

$$D_0 + D_1 \equiv D_0 + iD_1 \equiv -\sum_{\substack{0<n<F \\ n\equiv 3 \pmod 4}} \chi_F(n) \pmod{(1+i)Z[i]}\,. \tag{3.123}$$

Since

$$t_3(F) \equiv \sum_{\substack{0<n<F \\ n\equiv 3 \pmod 4 \\ (n,F)=1}} 1 \equiv -\sum_{\substack{0<n<F \\ n\equiv 3 \pmod 4}} \chi_F(n) \pmod{(1+i)Z[i]} \tag{3.124}$$

we deduce that

$$D_0 + D_1 \equiv t_3(F) \pmod{(1+i)Z[i]}\,.$$

This congruence and the fact that $(1+i)Z[i]\cap Z = 2Z$ imply that (3.114) holds in this case.

Next we treat the case $f = 8F$. We have $\chi = \chi_8\chi_F$, where χ_8 and χ_F are the characters $\pmod 8$ and $\pmod F$ respectively defined by

$$\begin{cases} \chi_8(1+Fn) = \chi(1+Fn)\,, & \text{for all } n\,, \\ \chi_F(1+8n) = \chi(1+8n)\,, & \text{for all } n\,. \end{cases} \tag{3.125}$$

As χ_8 is primitive, we have $\chi_8(5) = \chi_8(-3) = -1$. Hence we have

$$\chi_8(2F-n) = \begin{cases} \chi_8(n)\,, & \text{if } n \equiv F \pmod 4\,, \\ -\chi_8(n)\,, & \text{if } n \equiv -F \pmod 4\,. \end{cases} \tag{3.126}$$

Note also that

$$\chi_F(2F-n) = \chi_F(-n) = \chi_F(-1)\chi_F(n)\,. \tag{3.127}$$

Putting (3.126) and (3.127) together, we deduce

$$\chi(2F-n) = \begin{cases} \chi_F(-1)\chi(n)\,, & \text{if } n \equiv F \pmod 4\,, \\ -\chi_F(-1)\chi(n)\,, & \text{if } n \equiv -F \pmod 4\,. \end{cases} \tag{3.128}$$

Using (3.128) we obtain

$$\begin{aligned}\frac{1}{2}\sum_{0<n<f/4}\chi(n) &= \frac{1}{2}\sum_{0<n<F}(\chi(n)+\chi(2F-n))\\ &= \frac{1}{2}(1+\chi_F(-1))\sum_{\substack{0<n<F\\ n\equiv F \pmod 4}}\chi(n)\\ &\quad+\frac{1}{2}(1-\chi_F(-1))\sum_{\substack{0<n<F\\ n\equiv -F \pmod 4}}\chi(n),\end{aligned}$$

that is,

$$(3.129)\quad \frac{1}{2}\sum_{0<n<f/4}\chi(n)=\frac{1}{2}(1+\chi_F(-1))\sum_{0<n<F}\chi(n)-\chi_F(-1)\sum_{\substack{0<n<F\\ n\equiv -F \pmod 4}}\chi(n).$$

Since $\chi_F(-1)=1$ or -1, $\frac{1}{2}(1+\chi_F(-1))$ is an integer. Also we have

$$\sum_{0<n<F}\chi(n)\equiv\sum_{0<n<F}|\chi(n)|\equiv\sum_{\substack{0<n<F\\ (n,f)=1}}1=\frac{\phi(F)}{2}\equiv 0 \pmod{(1+i)Z[i]}.$$

Hence we have

$$\begin{aligned}\frac{1}{2}\sum_{0<n<f/4}\chi(n) &\equiv -\chi_F(-1)\sum_{\substack{0<n<F\\ n\equiv -F \pmod 4}}\chi(n) \pmod{(1+i)Z[i]}\\ &\equiv \sum_{\substack{0<n<F\\ n\equiv -F \pmod 4}}|\chi(n)| \pmod{(1+i)Z[i]},\end{aligned}$$

that is,

$$(3.130)\qquad \frac{1}{2}\sum_{0<n<f/4}\chi(n)\equiv t_{-F}(F) \pmod{(1+i)Z[i]}.$$

Now, modulo $(1+i)Z[i]$, we have

$$\begin{aligned}D_0+D_1 &\equiv D_0+D_1 i\\ &\equiv \frac{1}{2}((D_0-D_2)+(D_1-D_3)i)+\frac{1}{2}((D_0+D_2)+(D_1+D_3)i)\\ &\equiv \frac{1}{2}\sum_{0<n<f/4}\chi(n)+\frac{\phi(f)}{16}(1+i),\end{aligned}$$

that is, as $\phi(f)\equiv 0 \pmod{16}$,

$$(3.131)\qquad D_0+D_1\equiv\frac{1}{2}\sum_{0<n<f/4}\chi(n) \pmod{(1+i)Z[i]}.$$

From (3.130) and (3.131) we obtain (3.114).

Suppose now that $f = 16F$. We have $\chi = \chi_{16}\chi_F$, where χ_{16} and χ_F are the characters (mod 16) and (mod F) respectively defined by

$$\begin{aligned} \chi_{16}(1+Fn) &= \chi(1+Fn), \quad \text{for all } n, \\ \chi_F(1+16n) &= \chi(1+16n), \quad \text{for all } n. \end{aligned} \tag{3.132}$$

Note that

$$\chi_{16}(4F-n) = \begin{cases} \chi_{16}(3)\chi_{16}(n), & \text{if } n \equiv F \pmod 4, \\ \chi_{16}(3)^{-1}\chi_{16}(n), & \text{if } n \equiv -F \pmod 4, \end{cases} \tag{3.133}$$

and that

$$\chi_F(4F-n) = \chi_F(-1)\chi_F(n), \quad \text{for all } n. \tag{3.134}$$

Putting (3.133) and (3.134) together, we obtain

$$\chi(4F-n) = \begin{cases} \chi_F(-1)\chi_{16}(3)\chi(n), & \text{if } n \equiv F \pmod 4, \\ \chi_F(-1)\chi_{16}(3)^{-1}\chi(n), & \text{if } n \equiv -F \pmod 4. \end{cases} \tag{3.135}$$

Hence we have

$$\begin{aligned} \frac{1}{2}\sum_{0<n<4F}\chi(n) &= \frac{1}{2}\sum_{0<n<2F}(\chi(n)+\chi(4F-n)) \\ &= \frac{1}{2}(1+\chi_F(-1)\chi_{16}(3)) \sum_{\substack{0<n<2F \\ n\equiv F \pmod 4}} \chi(n) \\ &\quad + \frac{1}{2}(1+\chi_F(-1)\chi_{16}(3)^{-1}) \sum_{\substack{0<n<2F \\ n\equiv -F \pmod 4}} \chi(n). \end{aligned} \tag{3.136}$$

Since χ_{16} is a primitive character (mod 16), $\chi_{16}(9) = -1$, $\chi_{16}(3) = \pm i$, and thus

$$\chi_{16}(3)^{-1} = -\chi_{16}(3). \tag{3.137}$$

From (3.136) and (3.137) we obtain

$$\begin{aligned} \frac{1}{2}\sum_{0<n<4F}\chi(n) &= \frac{1}{2}(1+\chi_F(-1)\chi_{16}(3)) \sum_{0<n<2F}\chi(n) \\ &\quad - \chi_F(-1)\chi_{16}(3) \sum_{\substack{0<n<2F \\ n\equiv -F \pmod 4}} \chi(n). \end{aligned} \tag{3.138}$$

Next observe that

$$\chi_{16}(2F-n) = \begin{cases} \chi_{16}(n), & \text{if } n \equiv F \pmod 8, \\ -\chi_{16}(n), & \text{if } 5n \equiv F \pmod 8, \\ \chi_{16}(-3)\chi_{16}(n), & \text{if } 7n \equiv F \pmod 8, \\ \chi_{16}(-3)^{-1}\chi_{16}(n), & \text{if } 3n \equiv F \pmod 8. \end{cases} \tag{3.139}$$

From (3.139) we see that

$$\chi_{16}(2F-n) \equiv \begin{cases} \chi_{16}(n) \pmod{2Z[i]}, & \text{if } n \equiv F \pmod 4, \\ i\chi_{16}(n) \pmod{2Z[i]}, & \text{if } n \equiv -F \pmod 4. \end{cases} \tag{3.140}$$

Hence we have

$$\begin{aligned} \sum_{0<n<2F} \chi(n) &= \sum_{0<n<F} (\chi(n) + \chi(2F-n)) \\ &\equiv (1+\chi_F(-1)) \sum_{\substack{0<n<F \\ n\equiv F \pmod 4}} \chi(n) \\ &\quad + (1+i\chi_F(-1)) \sum_{\substack{0<n<F \\ n\equiv -F \pmod 4}} \chi(n) \pmod{2Z[i]}, \end{aligned}$$

that is,

$$\sum_{0<n<2F} \chi(n) \equiv (1+i) \sum_{\substack{0<n<F \\ n\equiv -F \pmod 4}} \chi(n) \pmod{2Z[i]}. \tag{3.141}$$

Also we have

(3.142)

$$\sum_{\substack{0<n<2F \\ n\equiv -F \pmod 4}} \chi(n) \equiv (1+i\chi_F(-1)) \sum_{\substack{0<n<F \\ n\equiv -F \pmod 4}} \chi(n) \equiv 0 \pmod{(1+i)Z[i]}.$$

From (3.138), (3.141) and (3.142), we obtain modulo $(1+i)Z[i]$,

$$\begin{aligned} \frac{1}{2}\sum_{0<n<4F} \chi(n) &\equiv \frac{1}{2}(1+\chi_F(-1)\chi_{16}(3))(1+i) \sum_{\substack{0<n<F \\ n\equiv -F \pmod 4}} \chi(n) \\ &\equiv \sum_{\substack{0<n<F \\ n\equiv -F \pmod 4}} \chi(n) \quad (\text{as } \chi_{16}(3) = \pm i), \end{aligned}$$

that is,

$$\frac{1}{2}\sum_{0<n<4F} \chi(n) \equiv t_{-F}(F). \tag{3.143}$$

From (3.143), as in the case $f = 8F$, we obtain

$$D_0 + D_1 \equiv t_{-F}(F) \pmod{(1+i)Z[i]},$$

from which (3.114) follows in this case. This completes the proof of (3.114).

Next we examine the value of $t_j(F)$, $j = 0, 1, 2, 3$, where F is an odd squarefree integer > 1. If $F = p$ (an odd prime) it is easy to show that

$$\begin{cases} t_0(p) = t_1(p) = t_2(p) = t_3(p) = \frac{1}{4}(p-1), & \text{if } p \equiv 1 \pmod 4, \\ t_0(p) = \frac{1}{4}(p-3),\ t_1(p) = \frac{1}{4}(p+1), & \\ \qquad t_2(p) = \frac{1}{4}(p+1),\ t_3(p) = \frac{1}{4}(p-3), & \text{if } p \equiv 3 \pmod 4. \end{cases} \tag{3.144}$$

If $F = pq$, where p and q are distinct odd primes, it is also easy to show that
(3.145)
$$\begin{cases} t_0(pq) = t_1(pq) = t_2(pq) = t_3(pq) = \frac{1}{4}(p-1)(q-1), & \text{if } p \text{ or } q \equiv 1 \pmod 4, \\ t_0(pq) = t_1(pq) = \frac{1}{4}(p-1)(q-1)+1, & \\ \quad t_2(pq) = t_3(pq) = \frac{1}{4}(p-1)(q-1)-1, & \text{if } p \equiv q \equiv 3 \pmod 4. \end{cases}$$

Equation (3.145) shows that

$$t_j(F) \equiv 0 \pmod 2, \qquad j = 0, 1, 2, 3,$$

when F is a product of two distinct odd primes.

Suppose now that F has more than two prime factors. We will prove that

$$t_j(F) \equiv 0 \pmod 2, \qquad j = 0, 1, 2, 3, \tag{3.146}$$

by induction on the number of prime factors of F. Let p be a prime factor of F and write $F = pF_1$, where F_1 has at least two distinct odd prime factors. We have

$$\begin{aligned} t_j(F) &= \sum_{\substack{0<n<F \\ (n,F)=1 \\ n\equiv j \pmod 4}} 1 = \sum_{\substack{0<n<F \\ (n,F_1)=1 \\ n\equiv j \pmod 4}} 1 - \sum_{\substack{0<n<F \\ (n,F_1)=1 \\ n\equiv 0 \pmod p \\ n\equiv j \pmod 4}} 1 \\ &= \sum_{k=0}^{p-1} \sum_{\substack{n=kF_1+1 \\ (n,F_1)=1 \\ n\equiv j \pmod 4}}^{(k+1)F_1} 1 - \sum_{\substack{0<m<F_1 \\ (m,F_1)=1 \\ pm\equiv j \pmod 4}} 1 \\ &= \sum_{k=0}^{p-1} \sum_{\substack{l=1 \\ (l,F_1)=1 \\ l\equiv j-kF_1 \pmod 4}}^{F_1} 1 - \sum_{\substack{0<m<F_1 \\ (m,F_1)=1 \\ m\equiv jp \pmod 4}} 1, \end{aligned}$$

that is,

$$\begin{aligned} t_j(F) = t_0(F_1) &\sum_{\substack{k=0 \\ k\equiv jF_1 \pmod 4}}^{p-1} 1 + t_1(F_1) \sum_{\substack{k=0 \\ k\equiv jF_1-F_1 \pmod 4}}^{p-1} 1 \\ &+ t_2(F_1) \sum_{\substack{k=0 \\ k\equiv jF_1-2F_1 \pmod 4}}^{p-1} 1 + t_3(F_1) \sum_{\substack{k=0 \\ k\equiv jF_1-3F_1 \pmod 4}}^{p-1} 1 - t_{jp}(F_1) \end{aligned} \tag{3.147}$$

from which (3.146) follows by induction.

Next, from (3.114) and (3.146), we obtain $D_0 + D_1 \equiv 0 \pmod 2$, if f is even, completing the proof of (3.91).

Finally we prove (3.92) when f is even. From (3.11), (3.13), (3.14), (3.90) and (3.91), we obtain

$$\begin{aligned}2h^*(K) &= (D_0 - D_2)^2 + (D_1 - D_3)^2 \\ &= \left(2D_0 - \frac{\phi(f)}{8}\right)^2 + \left(2D_1 - \frac{\phi(f)}{8}\right)^2 \\ &= 4(D_0^2 + D_1^2) - \frac{\phi(f)}{2}(D_0 + D_1) + \frac{(\phi(f))^2}{32} \\ &\equiv 4(D_0 + D_1) \pmod 8 \\ &\equiv 0 \pmod 8,\end{aligned}$$

completing the proof of (3.92) for f even.

The proof of Proposition 1 now follows from Lemmas 4, 5, 6, 7, 8 and the fact that $h^*(K) \equiv 1 \pmod 2$ if f has only one prime factor [14].

4. Proof of Proposition 2

The proof of Proposition 2 depends upon a number of lemmas.

Lemma 9 (Landau [11, pp. 27–28]). *For any real number $s > 1$ we have*

(4.1) $$\frac{1}{s-1} < \zeta(s) < 1 + \frac{1}{s-1}$$

and

(4.2) $$\frac{\zeta'(s)}{\zeta(s)} > -(s-1) - \frac{1}{s-1}.$$

Lemma 10 (see for example [1, pp. 55–56]). *For any positive integer N, we have*

(4.3) $$\left|\sum_{n=1}^{N} \frac{1}{n} - \log N - \gamma\right| \leq \frac{1}{N},$$

where

$$\gamma = 0.5772156649\ldots$$

is Euler's constant.

Lemma 11 (Polya-Landau inequality [13, 9]; see also [1, pp. 299–300]). *Let λ be a nonprincipal character* mod k *of conductor m. Then, for any positive integer n, we have*

(4.4)
$$\left|\sum_{r=1}^{n} \lambda(r)\right| \leq d(k/m)\frac{\sqrt{m}}{2\pi}\left(\log m + 2\log\log m + \frac{\pi}{\sqrt{m}} + 2\gamma + \frac{6\sqrt{m}\log m}{\sqrt{m}\log m - 1}\right),$$

where $d(l)$ denotes the number of positive integers dividing the positive integer l.

Lemma 12 (Tatuzawa [15, Lemma 5], Uchida [17, Lemma 2(i)]). *Let* λ *be a nonprincipal character* $(\bmod k)$. *Suppose that* N *is an integer such that*

$$\left|\sum_{r=1}^{n}\lambda(r)\right| \le N \tag{4.5}$$

for all positive integers n. *Then we have*

$$|L(s,\lambda)| \le \frac{|s|}{\sigma}\sum_{n=1}^{N}\frac{1}{n^{\sigma}}, \tag{4.6}$$

for $s=\sigma+it$, $\sigma>0$.

Lemma 13 (Landau [10, 11]). *Suppose that* $r>0$ *and* $s_0>1$, *and let* $f(z)$ *be analytic in the disk* $|z-s_0|\le r$. *Assume that* $f(s_0)\ne 0$, *and for some positive constant* M

$$|f(z)/f(s_0)| \le e^{M}, \quad \text{for } |z-s_0|\le r. \tag{4.7)(i}$$

Assume also that there is a constant E *such that*

$$E>2 \quad \text{on} \quad 0<s_0-1\le r/E. \tag{4.7)(ii}$$

Let s *and* r_1 *be real numbers such that*

$$1\le s\le s_0, \qquad f(s)\ne 0 \quad \text{and} \quad r/E<r_1\le r/2; \tag{4.8}$$

then

$$\Re\left(\frac{f'(s)}{f(s)}\right) \le \sum_{\rho}\Re\left(\frac{1}{s-\rho}\right) + \frac{2E(r_1E+r)}{(r_1E-r)^2}M, \tag{4.9}$$

where ρ *runs through all the zeros of* $f(z)$ (*counted according to multiplicity*) *such that* $|\rho-s_0|\le r_1$.

Proof. Set

$$g(z)=f(z)/\prod_{\rho}(z-\rho), \tag{4.10}$$

where ρ runs through all the zeros of $f(z)$ (counted according to multiplicity) in $|z-s_0|\le r_1$. Then $g(z)$ is analytic in $|z-s_0|\le r$ and has no zeros in $|z-s_0|\le r_1$. For $|z-s_0|=r$ we have (as $r_1\le r/2$)

$$\begin{aligned}\left|\frac{g(z)}{g(s_0)}\right| &= \left|\frac{f(z)}{f(s_0)}\right|\prod_{\rho}\frac{|s_0-\rho|}{|z-\rho|} \le \left|\frac{f(z)}{f(s_0)}\right|\prod_{\rho}\frac{r_1}{r-r_1}\\ &\le \left|\frac{f(z)}{f(s_0)}\right|\prod_{\rho}1 = \left|\frac{f(z)}{f(s_0)}\right| \le e^{M}.\end{aligned}$$

Then, by the Maximum Modulus Principle, we have

$$|g(z)/g(s_0)|\le e^{M}, \quad \text{for } |z-s_0|\le r_1. \tag{4.11}$$

Now there exists an analytic function $h(z)$ in $|z - s_0| \le r_1$ such that

$$(4.12) \qquad e^{h(z)} = g(z)/g(s_0), \qquad h(s_0) = 0, \qquad \Re(h(z)) \le M.$$

Set

$$(4.13) \qquad \phi(z) = h(z)/(2M - h(z)), \quad \text{for } |z - s_0| \le r_1.$$

The function $\phi(z)$ is analytic in $|z - s_0| \le r_1$ and such that $\phi(s_0) = 0$. Now let

$$(4.14) \qquad u = \Re(h), \qquad v = \operatorname{Im}(h),$$

so

$$(4.15) \qquad \phi(z) = \frac{u + iv}{(2M - u) - iv}, \qquad u \le M,$$

giving

$$|\phi(z)|^2 = \frac{u^2 + v^2}{(2M - u)^2 + v^2} \le 1,$$

as $(2M - u)^2 - u^2 = 4M(M - u) \ge 0$. Hence we have

$$(4.16) \qquad |\phi(z)| \le 1, \quad \text{for } |z - s_0| \le r_1.$$

By Schwarz's lemma [18, p. 189], we have

$$(4.17) \qquad |\phi(s)| \le \frac{1}{r_1}|s - s_0| \le \frac{s_0 - 1}{r_1} \le \frac{1}{E}\frac{r}{r_1} < 1.$$

Thus we have

$$\begin{aligned} |h(s)| &= \left| \frac{2M\phi(s)}{1 + \phi(s)} \right| \le \frac{2M|\phi(s)|}{1 - |\phi(s)|} \\ &\le 2M \cdot \frac{1}{E}\frac{r}{r_1} \cdot \frac{1}{1 - (1/E)(r/r_1)} = 2M\frac{r/r_1}{E - (r/r_1)}, \end{aligned}$$

and so for $|z - s_0| \le r_1$ we have

$$\begin{aligned} \Re(h(z) - h(s)) &= \Re(h(z)) - \Re(h(s)) \le \Re(h(z)) + |h(s)| \\ &\le M + 2M\frac{r/r_1}{E - r/r_1} = \left(\frac{E + r/r_1}{E - r/r_1}\right) M. \end{aligned}$$

Now set

$$(4.18) \quad \Psi(z) = \frac{(h(z) - h(s))}{2((E + r/r_1)/(E - r/r_1))M - (h(z) - h(s))}, \quad \text{for } |z - s_0| \le r_1.$$

$\psi(z)$ is analytic in $|z - s_0| \le r_1$ and

$$(4.19) \qquad |\Psi(z)| \le 1, \qquad \Psi(s) = 0.$$

Hence we have

$$(4.20) \qquad |\Psi(z)| \le 1, \quad \text{for } |z - s| \le r_1 - \varepsilon.$$

By Schwarz's lemma [18, p. 189] we obtain

$$|\Psi'(s)| = \left|\frac{h'(s)}{2((E+r/r_1)/(E-r/r_1))M}\right| \le \frac{1}{r_1-\varepsilon} \le \frac{1}{r_1 - r/E}, \tag{4.21}$$

so that

$$|h'(s)| \le \frac{2E(r_1E+r)}{(r_1E-r)^2}M. \tag{4.22}$$

Now

$$h'(s) = \frac{g'(s)}{g(s)} = \frac{f'(s)}{f(s)} - \sum_\rho \frac{1}{s-\rho}, \tag{4.23}$$

so

$$\Re\left(\frac{f'(s)}{f(s)} - \sum_\rho \frac{1}{s-\rho}\right) \le \left|\frac{f'(s)}{f(s)} - \sum_\rho \frac{1}{s-\rho}\right| \le \frac{2E(r_1E+r)}{(r_1E-r)^2}M,$$

giving

$$\Re\left(\frac{f'(s)}{f(s)}\right) \le \sum_\rho \Re\left(\frac{1}{s-\rho}\right) + \frac{2E(r_1E+r)}{(r_1E-r)^2}M. \tag{4.24}$$

This completes the proof of Lemma 13.

Lemma 14 [12, Lemma 1]. *Suppose that*

$$\sum_{m=0}^{n} a_m \cos m\phi \ge 0 \qquad (n \ge 1, a_m \ge 0) \tag{4.25}$$

for all real ϕ. *Then we have*

$$a_0 \frac{L'(\sigma,\lambda_0)}{L(\sigma,\lambda_0)} + \frac{1}{2}\sum_{m=1}^{n} a_m \Re\left(\frac{L'(\sigma,\lambda^m)}{L(\sigma,\lambda^m)} + \frac{L'(\sigma,\bar{\lambda}^m)}{L(\sigma,\bar{\lambda}^m)}\right) \le 0, \tag{4.26}$$

for $\sigma > 1$ *and any character* λ. (λ_0 *denotes the principal character.*)

Lemma 15 [6, pp. 146–150; 12, Lemma 2]. *Let* λ *be a nonprincipal character* $(\bmod k)$ *and let* $y = 0$ *or* 1 *according as* λ *is even or odd. Then for* $\sigma \ge 1/2$

$$\begin{aligned} &\Re\left(\frac{L'(s,\lambda)}{L(s,\lambda)}\right) + \Re\left(\frac{L'(s,\bar{\lambda})}{L(s,\bar{\lambda})}\right) \\ &\quad \ge \log\pi - \log k - \Re\frac{\Gamma'(\frac{1}{2}(s+y))}{\Gamma(\frac{1}{2}(s+y))} + \Re\sum_\rho\left(\frac{1}{s-\rho} + \frac{1}{s-1+\rho}\right), \end{aligned} \tag{4.27}$$

where $s = \sigma + it$ *and in the last sum* $\rho = \beta + i\gamma$ *runs through all the zeros of* $L(s,\lambda)$ *in the strip* $0 < \beta < 1$.

From this point on in this section we will make use of A, B, C, D, a, b, c as defined in (1.2)–(1.8). We assume that the conductor f satisfies (1.1) and we set

(4.28) $$s_0 = 1 + \frac{1}{a\log f}, \qquad s_1 = 1 + \frac{1}{b\log f},$$

(4.29) $$s = 1 + \frac{x}{a\log f}, \qquad 0 \le x \le 1,$$

so that

(4.30) $$1 \le s \le s_0 < s_1, \qquad s_0 > 1.$$

We also note that

(4.31) $$\log f \ge \log A$$

and

(4.32) $$\log\log f \le \left(\frac{\log\log A}{\log A}\right)\log f.$$

Inequalities (4.31) and (4.32) will be used on many occasions without comment. Set $L_1(s) = L(s, \chi)L(s, \chi^3)$, $L_2(s) = L(s, \chi^2)$.

Lemma 16. $\log\zeta(s_0) + \log L_1(s_0) + \log L_2(s_0) \ge 0$.

Proof. As $s_0 > 1$ we have

(4.33) $$\log\zeta(s_0) = -\sum_p \log\left(1 - \frac{1}{p^{s_0}}\right) = \sum_p \sum_{n=1}^{\infty} \frac{1}{np^{ns_0}}$$

and, for $j = 1, 2, 3$,

(4.34) $$\log L(s_0, \chi^j) = -\sum_p \log\left(1 - \frac{\chi^j(p)}{p^{s_0}}\right) = \sum_p \sum_{n=1}^{\infty} \frac{\chi(p)^{nj}}{np^{ns_0}},$$

where χ is the character defined in §2 and p runs through all primes. Adding (4.33) and (4.34) we obtain

$$\begin{aligned}\log\zeta(s_0) + \log L_1(s_0) + \log L_2(s_0) &= \log\zeta(s_0) + \sum_{j=1}^{3}\log L(s_0, \chi^j)\\ &= \sum_p \sum_{n=1}^{\infty} \frac{(1 + \chi(p)^n + \chi(p)^{2n} + \chi(p)^{3n})}{np^{ns_0}} \ge 0\end{aligned}$$

as $1 + \chi(p)^n + \chi(p)^{2n} + \chi(p)^{3n} = 0$ or 4.

Lemma 17 [17, Equation (3)].

$$-\log L_1(1) < \log a + \log\log f + \frac{1}{a\log f} + \log L_2(s_0) + \int_1^{s_0} \frac{L_1'(s)}{L_1(s)}\,ds.$$

Proof. We have

(4.35) $$-\log L_1(1) = -\log L_1(s_0) + \int_1^{s_0} \frac{L_1'(s)}{L_1(s)}\,ds.$$

By Lemma 16, we have

$$\log \zeta(s_0) + \log L_1(s_0) + \log L_2(s_0) \geq 0, \tag{4.36}$$

so that

$$-\log L_1(1) \leq \log \zeta(s_0) + \log L_2(s_0) + \int_1^{s_0} \frac{L_1'(s)}{L_1(s)}\, ds. \tag{4.37}$$

Now, by Lemma 9 and (4.28), we have

$$\zeta(s_0) < 1 + 1/(s_0 - 1) = 1 + a\log f. \tag{4.38}$$

Hence we have

$$\begin{aligned} \log \zeta(s_0) &< \log(1 + a\log f) \\ &= \log\left(a\log f\left(1 + \frac{1}{a\log f}\right)\right) \\ &= \log a + \log\log f + \log\left(1 + \frac{1}{a\log f}\right) \\ &< \log a + \log\log f + \frac{1}{a\log f}. \end{aligned}$$

This completes the proof of Lemma 17.

Lemma 18. *For any positive integer* n, *we have*

$$\left|\sum_{r=1}^{n} \chi(r)\right| < [B\sqrt{f}\log f], \tag{4.39}$$

where B *is defined in* (1.3).

Proof. By Lemma 11, as χ is a primitive nonprincipal character mod f, we have

$$\begin{aligned} 1 + \left|\sum_{r=1}^{n} \chi(r)\right| &\leq 1 + \frac{\sqrt{f}}{2\pi}\left(\log f + 2\log\log f + \frac{6\sqrt{f}\log f}{\sqrt{f}\log f - 1} + 2\gamma + \frac{\pi}{\sqrt{f}}\right) \\ &\leq \frac{3\sqrt{f}\log f}{2\sqrt{A}\log A} + \frac{\sqrt{f}\log f}{2\pi} + \frac{1}{\pi}\frac{\log\log A}{\log A}\sqrt{f}\log f \\ &\quad + \frac{3}{\pi}\frac{\sqrt{f}\log f}{\log A - 1/\sqrt{A}} + \frac{\gamma}{\pi}\sqrt{f}\frac{\log f}{\log A} \\ &= B\sqrt{f}\log f, \end{aligned}$$

so

$$\left|\sum_{r=1}^{n} \chi(r)\right| \leq B\sqrt{f}\log f - 1 < [B\sqrt{f}\log f],$$

as asserted.

Lemma 19 [17, Lemma 2(ii)]. *For* z *a complex number satisfying*

$$|z - s_0| \leq c, \tag{4.40}$$

we have

$$|L(z,\chi)| < \frac{1}{c\sqrt{1-c^2}} B^c f^{c/2} (\log f)^c. \tag{4.41}$$

Proof. By Lemmas 12 and 18, we have

$$|L(z,\chi)| \le \frac{|z|}{\sigma_1} \sum_{n=1}^{N} \frac{1}{n^{\sigma_1}}, \tag{4.42}$$

where $z = \sigma_1 + it_1$, $\sigma_1 > 0$, and $N = [B\sqrt{f}\log f]$. Now for $|z - s_0| \le c$ an easy calculation shows that

$$\frac{|z|}{\sigma_1} \le \frac{1}{\sqrt{1-c^2}}. \tag{4.43}$$

Further we have

$$\begin{aligned} \sum_{n=1}^{N} \frac{1}{n^{\sigma_1}} &\le \sum_{n=1}^{N} \frac{1}{n^{1-c}} = 1 + \sum_{n=2}^{N} \frac{1}{n^{1-c}} \\ &\le 1 + \int_1^N \frac{dx}{x^{1-c}} = 1 + \frac{N^c}{c} - \frac{1}{c} < \frac{N^c}{c}, \end{aligned}$$

that is,

$$\sum_{n=1}^{N} \frac{1}{n^{\sigma_1}} \le \frac{1}{c} B^c f^{c/2} (\log f)^c. \tag{4.44}$$

Hence, for $|z - s_0| \le c$, we have

$$|L(z,\chi)| < \frac{1}{c\sqrt{1-c^2}} B^c f^{c/2} (\log f)^c,$$

as required.

Lemma 20 [17, Lemma 3(i)]. *Let ρ be a zero of $L(z,\chi)$. For $1 \le s < s_1$ we have*

$$\Re\left(\frac{1}{s-\rho}\right) \le \frac{1}{1-(s_1-s)\Re(1/(s_1-\rho))} \Re\left(\frac{1}{s_1-\rho}\right). \tag{4.45}$$

Proof. Set $\rho = \sigma + it$. As ρ is a zero of $L(z,\chi)$ we have $\sigma < 1$. We have

$$\Re\left(\frac{1}{s-\rho}\right) = \Re\left(\frac{1}{(s-\sigma)-it}\right) = \Re\left(\frac{(s-\sigma)+it}{(s-\sigma)^2+t^2}\right),$$

that is,

$$\Re\left(\frac{1}{s-\rho}\right) = \frac{s-\sigma}{(s-\sigma)^2+t^2}. \tag{4.46}$$

Now set

$$X = \Re\left(\frac{1}{s_1-\rho}\right) = \frac{s_1-\sigma}{(s_1-\sigma)^2+t^2} > 0. \tag{4.47}$$

Then we obtain

$$\Re\left(\frac{1}{s-\rho}\right) = \frac{s-\sigma}{(s-\sigma)^2-(s_1-\sigma)^2+(s_1-\sigma)/X} = \frac{s-\sigma}{2(s-s_1)(s-\sigma)-(s-s_1)^2+(s_1-\sigma)/X},$$

that is,

$$\Re\left(\frac{1}{s-\rho}\right) = \frac{(s-\sigma)X}{2(s-s_1)(s-\sigma)X-(s-s_1)^2X+(s_1-\sigma)}. \tag{4.48}$$

Now clearly we have

$$\frac{t^2}{(s_1-\sigma)^2+t^2} \geq 0,$$

so

$$1-\frac{(s_1-\sigma)^2}{(s_1-\sigma)^2+t^2} \geq 0,$$

giving

$$1-(s_1-\sigma)X \geq 0. \tag{4.49}$$

Multiplying (4.49) by $(s_1-s)>0$, we obtain

$$(s-s_1)(s_1-\sigma)X+(s_1-s) \geq 0,$$

giving

$$(s-s_1)(s-\sigma)X-(s-s_1)^2X+(s_1-s) \geq 0,$$

and so

$$2(s-s_1)(s-\sigma)X-(s-s_1)^2X+(s_1-\sigma) \geq (s-s_1)(s-\sigma)X+(s-\sigma). \tag{4.50}$$

Now $s-\sigma>0$ and

$$(s-s_1)X+1 > (\sigma-s_1)X+1 \geq 0,$$

so we have

$$(s-s_1)(s-\sigma)X+(s-\sigma) > 0. \tag{4.51}$$

Thus we have, from (4.50) and (4.51),

$$\frac{1}{(s-s_1)(s-\sigma)X+(s-\sigma)} \geq \frac{1}{2(s-s_1)(s-\sigma)X-(s-s_1)^2X+(s_1-\sigma)} \tag{4.52}$$

and so by (4.48) and (4.52) we obtain

$$\Re\left(\frac{1}{s-\rho}\right) \leq \frac{(s-\sigma)X}{(s-s_1)(s-\sigma)X+(s-\sigma)} = \frac{X}{(s-s_1)X+1}$$

as required.

Lemma 21 [17, Lemma 3(ii)]. *Assume that*

$$|L_1(z)/L_1(s_0)| \leq e^M, \quad \textit{where } M>0, \tag{4.53}$$

for $|z - s_0| \le c$. *Then, we have*

$$\frac{L_1'(s)}{L_1(s)} \le \sum_\rho \Re\left(\frac{1}{s-\rho}\right) + \frac{4a(\log A)(ac\log A + 2)}{(ac\log A - 2)^2} M, \tag{4.54}$$

where ρ *runs through all the zeros of* $L_1(z)$ *(counted according to multiplicity) in* $|\rho - s_0| \le c/2$.

Proof. In Lemma 13 we take

$$f(z) = L_1(z), \qquad r = c, \qquad r_1 = c/2, \qquad E = ac\log A. \tag{4.55}$$

It is easily verified that the hypotheses of Lemma 13 are satisfied and assertion (4.54) of Lemma 21 follows as $L_1'(s)/L_1(s)$ is real.

Lemma 22. *If* $h^*(K) \equiv 2 \pmod 4$ *then we have*

$$|L_2(s_0)| < \left(\frac{1}{2} + \frac{\log\log A}{\log A} + \frac{\gamma}{\log A} + \frac{1}{(V\sqrt{A}\log A - 1)\log A} + \frac{\log V}{\log A}\right)\log f,$$

where

$$V = \frac{1}{\pi\sqrt{2}} + \frac{\sqrt{2}\log\log A}{\pi\log A} + \frac{3}{2\pi((1/\sqrt{8})\log(A/8) - 1/\sqrt{A})} + \frac{\gamma}{\pi\log A} + \frac{2}{\sqrt{A}\log A}.$$

Proof. For $x \ge 5$ set

$$M(x) = \frac{\sqrt{x}\log x}{2\pi} + \frac{\sqrt{x}\log\log x}{\pi} + \frac{3x\log x}{\pi(\sqrt{x}\log x - 1)} + \frac{\gamma}{\pi}\sqrt{x} + \frac{1}{2}. \tag{4.56}$$

It is easy to check that $M(x)$ is an increasing function of x for $x \ge 5$. By Proposition 1, as $h^*(K) \equiv 2 \pmod 4$, we have $f = 16p$ $(p \equiv 3, 5 \pmod 8)$; $8p$ $(p \equiv 5 \pmod 8)$; or pq $((p/q) = -1)$, where p and q denote distinct odd primes. Now χ^2 is a nonprincipal character mod f of conductor m, wher m is given by (see [6, (3.4) and Theorem 5])

$$m = \begin{cases} 8 \text{ or } 8p, & \text{if } f = 16p, \\ p, & \text{if } f = 8p, \\ p, q, \text{ or } pq, & \text{if } f = pq. \end{cases}$$

Thus $d(f/m) = 1, 2$ and 4 and so, by Lemma 11, we have for any positive integer n (noting that $8 < f/8$ so that $M(8) < M(f/8)$)

$$\left|\sum_{r=1}^n \chi^2(r)\right| \le \max(M(f), 2M(f/2), 4M(f/8)). \tag{4.57}$$

Observe that the function $(\sqrt{x}\log x)/(\sqrt{x}\log x - 1)$ is decreasing when $x \ge e$. Therefore

$$4\frac{(f/8)\log(f/8)}{\sqrt{f/8}\log(f/8) - 1} > \max\left(2\frac{(f/2)\log(f/2)}{\sqrt{f/2}\log(f/2) - 1}, \frac{f\log f}{\sqrt{f}\log f - 1}\right). \tag{4.58}$$

Set $g(x) = (\sqrt{x}\log x)/2\pi + \gamma\sqrt{x}/\pi$. Observe that

$$\frac{\sqrt{f}\log f}{\pi\sqrt{2}} + \frac{\gamma}{\pi}\sqrt{f} - g(f) = \frac{\sqrt{f}\log f}{\pi\sqrt{2}}\left(1 - \frac{1}{\sqrt{2}}\right) > 0,$$

$$\frac{\sqrt{f}\log f}{\pi\sqrt{2}} + \frac{\gamma}{\pi}\sqrt{f} - 2g\left(\frac{f}{2}\right) = \frac{\sqrt{f}}{\pi\sqrt{2}}(\log 2 + \sqrt{2}\gamma - 2\gamma) > 0,$$

and

$$\frac{\sqrt{f}\log f}{\pi\sqrt{2}} + \frac{\gamma}{\pi}\sqrt{f} - 4g\left(\frac{f}{8}\right) = \frac{\sqrt{f}}{\pi\sqrt{2}}(\log 8 + \sqrt{2}\gamma - 2\gamma) > 0.$$

These inequalities and statement (4.58) imply that

$$\frac{\sqrt{f}\log f}{\pi\sqrt{2}} + \frac{\gamma}{\pi}\sqrt{f} + \frac{\sqrt{2}}{\pi}\sqrt{f}\log\log f + \frac{3f\log(f/8)}{2\pi(\sqrt{f/8}\log(f/8) - 1)} + 2$$
$$\geq \max\left(M(f), 2M\left(\frac{f}{2}\right), 4M\left(\frac{f}{8}\right)\right).$$

Combining this with (4.57) yields

$$\left|\sum_{r=1}^{n}\chi^2(r)\right| \leq \left(\frac{1}{\pi\sqrt{2}} + \frac{\sqrt{2}\log\log A}{\log A} + \frac{3}{2\pi(\frac{1}{\sqrt{8}}\log\frac{A}{8} - \frac{1}{\sqrt{A}})}\right.$$
$$\left. + \frac{\gamma}{\pi\log A} + \frac{2}{\sqrt{A}\log A}\right)\sqrt{f}\log f,$$

so that (as $\chi^2(r) = 0, \pm 1$)

$$\left|\sum_{r=1}^{n}\chi^2(r)\right| \leq [V\sqrt{f}\log f],$$

where V is defined in the statement of Lemma 22.

Then, by the definition of $L_2(s)$, Lemma 10, and Lemma 12 with $\lambda = \chi^2$, $k = f$, $N = [V\sqrt{f}\log f]$, we obtain

$$\begin{aligned}
|L_2(s_0)| = |L(s_0, \chi^2)| &\leq \sum_{n=1}^{N}\frac{1}{n^{s_0}} \\
&\leq \sum_{n=1}^{N}\frac{1}{n} \quad (\text{as } s_0 > 1) \\
&\leq \log N + \gamma + \frac{1}{N} \\
&< \log V + \frac{1}{2}\log f + \log\log f + \gamma + \frac{1}{V\sqrt{f}\log f - 1} \\
&\leq \left(\frac{1}{2} + \frac{\log\log A}{\log A} + \frac{\gamma}{\log A} + \frac{1}{(V\sqrt{A}\log A - 1)\log A} + \frac{\log V}{\log A}\right)\log f
\end{aligned}$$

as required.

Lemma 23. *For* $|z - s_0| \leq c$ *we have*

$$\log|L_1(z)/L_1(s_0)| \leq C \log f, \tag{4.59}$$

where C *is defined in* (1.7). *This result shows that the assumed inequality in Lemma* 21 *holds with*

$$M = C \log f. \tag{4.60}$$

Proof. Appealing to Lemmas 9, 16, 19, 22, with

$$T = \frac{1}{2} + \frac{\log\log A}{\log A} + \frac{\gamma}{\log A} + \frac{1}{(V\sqrt{A} - 1)\log A} + \frac{\log V}{\log A},$$

we obtain

$$\begin{aligned}
\log\left|\frac{L_1(z)}{L_1(s_0)}\right| &= \log|L(z, \chi)| + \log|L(z, \chi^3)| - \log|L_1(s_0)| \\
&\leq \log|L(z, \chi)| + \log|L(z, \chi^3)| + \log\zeta(s_0) + \log L_2(s_0) \\
&< 2\left(c\log B + (c/2)\log f + c\log\log f - \log(c\sqrt{1 - c^2})\right) \\
&\quad + \log\left(1 + \frac{1}{s_0 - 1}\right) + \log\log f + \log T \\
&= c\log f + (2c + 1)\log\log f + 2c\log B \\
&\quad - 2\log(c\sqrt{1 - c^2}) + \log(1 + a\log f) + \log T.
\end{aligned}$$

Now, as

$$\begin{aligned}
\log(1 + a\log f) &= \log(a\log f) + \log\left(1 + \frac{1}{a\log f}\right) \\
&\leq \log a + \log\log f + \frac{1}{a\log f},
\end{aligned}$$

we have

$$\begin{aligned}
\log\left|\frac{L_1(z)}{L_1(s_0)}\right| &< c\log f + 2(c + 1)\log\log f + \log a \\
&\quad + \frac{1}{a\log f} + 2c\log B - 2\log(c\sqrt{1 - c^2}) + \log T \\
&\leq c\log f + 2(c + 1)\frac{\log\log A}{\log A}\log f + \frac{1}{a(\log A)^2}\log f \\
&\quad + (\log a + 2c\log B - 2\log(c\sqrt{1 - c^2}) + \log T) \\
&\leq C\log f,
\end{aligned}$$

as required.

Lemma 24. *For* χ *the complex odd quartic primitive character* (mod f) *defined in* §2 *and* $1 < \sigma < 2$, *we have*

$$\frac{L'(\sigma, \chi^2)}{L(\sigma, \chi^2)} > -\frac{1}{2}\log f + 0.78. \tag{4.61}$$

Proof. Taking $\lambda = \chi^2$ in Lemma 15 we obtain, as $y = 0$ and $\bar{\lambda} = \bar{\chi}^2 = \chi^2$, for $\sigma \geq \frac{1}{2}$

(4.62)
$$2\frac{L'(\sigma, \chi^2)}{L(\sigma, \chi^2)} \geq \log \pi - \log f - \frac{\Gamma'(\sigma/2)}{\Gamma(\sigma/2)} + \Re \sum_{\substack{\rho \\ L(\rho, \chi^2)=0 \\ 0<\Re\rho<1}} \left(\frac{1}{\sigma - \rho} + \frac{1}{\sigma - 1 + \rho}\right).$$

Now, for $\sigma \geq 1$ and $0 < \Re\rho < 1$, we have

$$\Re\left(\frac{1}{\sigma - \rho}\right) = \frac{\sigma - \Re\rho}{|\sigma - \rho|^2} > 0 \tag{4.63}$$

and

$$\Re\left(\frac{1}{\sigma - 1 + \rho}\right) = \frac{\sigma - 1 + \Re\rho}{|\sigma - 1 + \rho|^2} > 0, \tag{4.64}$$

so that

$$\frac{L'(\sigma, \chi^2)}{L(\sigma, \chi^2)} \geq -\frac{1}{2}\log f + \frac{1}{2}\log \pi - \frac{1}{2}\frac{\Gamma'(\sigma/2)}{\Gamma(\sigma/2)}. \tag{4.65}$$

Now it is known [2, p. 17, equation (2.10)] that for real $x > 0$

$$\frac{\Gamma'(x)}{\Gamma(x)} < -0.57 - \frac{1}{x} + x\sum_{n=1}^{\infty} \frac{1}{n(n+x)}. \tag{4.66}$$

Hence, for $\frac{1}{2} < x < 1$, we have

$$\begin{aligned}
\frac{\Gamma'(x)}{\Gamma(x)} &< -0.57 - 1 + \frac{x}{1+x} + x\sum_{n=2}^{\infty}\frac{1}{n^2} \\
&= -1.57 + \frac{x}{1+x} + x\left(\frac{\pi^2}{6} - 1\right) \\
&< -1.57 + 0.5 + \frac{\pi^2}{6} - 1 \\
&< -2.07 + 1.65 = -0.42,
\end{aligned}$$

that is,

$$\Gamma'(\sigma/2)/\Gamma(\sigma/2) < -0.42, \quad \text{for } 1 < \sigma < 2. \tag{4.67}$$

Thus we obtain

$$\frac{L'(\sigma, \chi^2)}{L(\sigma, \chi^2)} > -\frac{1}{2}\log f + 0.57 + 0.21,$$

that is,

$$\frac{L'(\sigma, \chi^2)}{L(\sigma, \chi^2)} > -\frac{1}{2}\log f + 0.78, \quad \text{for } 1 < \sigma < 2. \tag{4.68}$$

Lemma 25 (see [17, p. 494]). *For any real numbers* s_0 *and* s_1 *such that*

(4.69) $$1 < s_0, \qquad 1 < s_1 < 2,$$

we have

(4.70) $$\sum_\rho \Re\left(\frac{1}{s_1-\rho}\right) < \frac{1}{s_1-1} + \frac{3}{2}\log f,$$

where ρ *runs through all the zeros of* $L_1(z)$ *counted according to multiplicity in* $|\rho - s_0| \le c/2$.

Proof. Since

$$2 + 3\cos\phi + 2\cos 2\phi + \cos 3\phi = 4\cos^2\phi(1+\cos\phi) \ge 0,$$

for all real ϕ, we may take

$$a_0 = 2, \qquad a_1 = 3, \qquad a_2 = 2, \qquad a_3 = 1, \qquad n = 3,$$

and $\lambda = \chi$, in Lemma 14 to obtain, for $\sigma > 1$,

$$2\frac{L'(\sigma,\chi_0)}{L(\sigma,\chi_o)} + \frac{3}{2}\Re\left(\frac{L'(\sigma,\chi)}{L(\sigma,\chi)} + \frac{L'(\sigma,\overline{\chi})}{L(\sigma,\overline{\chi})}\right)$$
$$+\Re\left(\frac{L'(\sigma,\chi^2)}{L(\sigma,\chi^2)} + \frac{L'(\sigma,\overline{\chi}^2)}{L(\sigma,\overline{\chi}^2)}\right) + \frac{1}{2}\Re\left(\frac{L'(\sigma,\chi^3)}{L(\sigma,\chi^3)} + \frac{L'(\sigma,\overline{\chi}^3)}{L(\sigma,\overline{\chi}^3)}\right) \le 0,$$

that is, as $\chi^3 = \overline{\chi}$, $\chi^2 = \overline{\chi}^2$, $\chi = \overline{\chi}^3$,

$$2\frac{L'(\sigma,\chi_o)}{L(\sigma,\chi_0)} + 2\Re\left(\frac{L'(\sigma,\chi)}{L(\sigma,\chi)} + \frac{L'(\sigma,\overline{\chi})}{L(\sigma,\overline{\chi})}\right) + 2\Re\left(\frac{L'(\sigma,\chi^2)}{L(\sigma,\chi^2)}\right) \le 0,$$

equivalently

(4.71) $$\frac{L'(\sigma,\chi_0)}{L(\sigma,\chi_0)} + \frac{L_1'(\sigma)}{L_1(\sigma)} + \frac{L'(\sigma,\chi^2)}{L(\sigma,\chi^2)} \le 0.$$

Then, appealing to Lemma 9, Lemma 24 and (4.71), we obtain, for $1 < \sigma < 2$,

$$\begin{aligned}\frac{L_1'(\sigma)}{L_1(\sigma)} &\le -\frac{L'(\sigma,\chi_0)}{L(\sigma,\chi_0)} - \frac{L'(\sigma,\chi^2)}{L(\sigma,\chi^2)}\\ &< -\frac{\zeta'(\sigma)}{\zeta(\sigma)} + \frac{1}{2}\log f - 0.78\\ &< \frac{1}{\sigma-1} + (\sigma-1) + \frac{1}{2}\log f - 0.78\\ &< \frac{1}{\sigma-1} + 1 + \frac{1}{2}\log f - 0.78,\end{aligned}$$

that is,

(4.72) $$\frac{L_1'(\sigma)}{L_1(\sigma)} < \frac{1}{\sigma-1} + \frac{1}{2}\log f + 0.22.$$

Next, appealing to Lemma 15 and noting that $y = 1$ as $\lambda = \chi$ $(\neq \chi_0)$ is odd, we have for $\sigma \geq \frac{1}{2}$

(4.73)
$$\frac{L_1'(\sigma)}{L_1(\sigma)} \geq \log \pi - \log f - \frac{\Gamma'(\frac{1}{2}(\sigma+1))}{\Gamma(\frac{1}{2}(\sigma+1))} + \Re \sum_{\substack{\rho \\ L(\rho,\chi)=0 \\ 0<\Re\rho<1}} \left(\frac{1}{\sigma-\rho} + \frac{1}{\sigma-(1-\rho)}\right).$$

Putting (4.72) and (4.73) together, we obtain for $1 < \sigma < 2$

$$(4.74) \qquad \Re \sum_{\substack{\rho \\ L(\rho,\chi)=0 \\ 0<\Re\rho<1}} \left(\frac{1}{\sigma-\rho} + \frac{1}{\sigma-(1-\rho)}\right) < \frac{1}{\sigma-1} + \frac{3}{2}\log f + 0.22 - \log \pi + \frac{\Gamma'(\frac{1}{2}(\sigma+1))}{\Gamma(\frac{1}{2}(\sigma+1))}.$$

Set $x = \frac{1}{2}(\sigma+1)$. As $1 < \sigma < 2$ we have $1 < x < 3/2$. Then, for $1 < x < 3/2$, we have

$$\begin{aligned} x\sum_{n=1}^{\infty} \frac{1}{n(n+x)} &= \frac{x}{1+x} + x\sum_{n=2}^{\infty}\frac{1}{n(n+x)} \\ &< \frac{x}{1+x} + x\sum_{n=2}^{\infty}\frac{1}{n(n+1)} \\ &< \frac{3/2}{1+3/2} + \frac{3}{2}\cdot\frac{1}{2} = 1.35, \end{aligned}$$

and so, by (4.66), we obtain

$$(4.75) \qquad \frac{\Gamma'(\frac{1}{2}(\sigma+1))}{\Gamma(\frac{1}{2}(\sigma+1))} < -0.57 - 0.66 + 1.35 = 0.12.$$

Using the estimate (4.75) in (4.74), we obtain

$$\begin{aligned} \Re \sum_{\substack{\rho \\ L(\rho,\chi)=0 \\ 0<\Re\rho<1}} \left(\frac{1}{\sigma-\rho} + \frac{1}{\sigma-(1-\rho)}\right) &< \frac{1}{\sigma-1} + \frac{3}{2}\log f + 0.22 - 1.14 + 0.12 \\ &= \frac{1}{\sigma-1} + \frac{3}{2}\log f - 0.8, \end{aligned}$$

that is,

$$(4.76) \qquad \Re \sum_{\substack{\rho \\ L(\rho,\chi)=0 \\ 0<\Re\rho<1}} \left(\frac{1}{\sigma-\rho} + \frac{1}{\sigma-(1-\rho)}\right) < \frac{1}{\sigma-1} + \frac{3}{2}\log f.$$

The functional equation for Dirichlet L-functions [1, p. 263] implies that

(4.77) if $z \in \mathbf{C}$ and z is not an integer, then $L(1-z,\overline{\chi}) = 0 \Leftrightarrow L(z,\chi) = 0$.

Note that $\Re\rho \geq s_0 - |\Re\rho - s_0| \geq s_0 - |\rho - s_0|$ for all $\rho \in \mathbf{C}$. Therefore, if $|\rho - s_0| \leq c/2$, then $\Re\rho \geq s_0 - c/2 > 1/2$. This shows that

$$\{\rho : |\rho - s_0| \leq c/2\} \subseteq \{\rho : \Re\rho > 1/2\}. \tag{4.78}$$

Observe that

$$\begin{aligned}
&\Re \sum_{\substack{\rho \\ L(\rho,\chi)=0 \\ 0<\Re\rho<1}} \left(\frac{1}{\sigma-\rho} + \frac{1}{\sigma-(1-\rho)}\right) \\
&= \Re\left(\sum_{\substack{\rho \\ L(\rho,\chi)=0 \\ 0<\Re\rho<1}} \frac{1}{\sigma-\rho} + \sum_{\substack{\rho \\ L(\rho,\bar{\chi})=0 \\ 0<\Re\rho<1}} \frac{1}{\sigma-\rho}\right) \quad \text{(by 4.77)} \\
&= \Re\left(\sum_{\substack{\rho \\ L_1(\rho)=0 \\ 0<\Re\rho<1}} \frac{1}{\sigma-\rho}\right).
\end{aligned}$$

Note also that, by (4.63) and (4.78),

$$\Re\left(\sum_{\substack{\rho \\ L_1(\rho)=0 \\ 0<\Re\rho<1}} \frac{1}{\sigma-\rho}\right) \geq \Re\left(\sum_{\substack{\rho \\ L_1(\rho)=0 \\ |\rho-s_0|\leq c/2 \\ \Re\rho<1}} \frac{1}{\sigma-\rho}\right) = \Re\left(\sum_{\substack{\rho \\ L_1(\rho)=0 \\ |\rho-s_0|\leq c/2}} \frac{1}{\sigma-\rho}\right),$$

since $L_1(z)$ has no zeros with $\Re z \geq 1$. Thus

$$\Re \sum_{\substack{\rho \\ L(\rho,\chi)=0 \\ 0<\Re\rho<1}} \left(\frac{1}{\sigma-\rho} + \frac{1}{\sigma-(1-\rho)}\right) \geq \Re\left(\sum_{\substack{\rho \\ L_1(\rho)=0 \\ |\rho-s_0|\leq c/2}} \frac{1}{\sigma-\rho}\right).$$

This inequality and (4.76) establish (4.70).

Lemma 26 (see [17, p. 494]).

$$\frac{L_1'(s)}{L_1(s)} \leq \frac{(b+3/2)a\log f}{a(1/2-3/4b)+x(b/2+3/4)} + \frac{4a(\log A)(ac\log A+2)}{(ac\log A-2)^2} C\log f.$$

Proof. For $s_1 = 1 + 1/(b\log f)$, where b is defined in (1.4), we have $s_1 > 1$. Also, as $b\log f > \log f \geq \log 64 > 4.1 > 1$, we have $s_1 < 2$. Then, appealing to Lemma 25, we have

$$\sum_{\substack{\rho \\ L_1(\rho)=0 \\ |\rho-s_0|\leq c/2}} \Re\left(\frac{1}{s_1-\rho}\right) < \frac{1}{s_1-1} + \frac{3}{2}\log f = b\log f + \frac{3}{2}\log f,$$

that is,

$$\sum_{\substack{\rho \\ L_1(\rho)=0 \\ |\rho - s_0| \le c/2}} \Re\left(\frac{1}{s_1-\rho}\right) < \left(b+\frac{3}{2}\right)\log f. \tag{4.79}$$

If ρ is a zero of $L_1(z)$ in $|z-s_0| \le c/2$, so is $\overline{\rho}$, and thus, from (4.79), we deduce

$$\Re\left(\frac{1}{s_1-\rho}\right) < \left(\frac{b}{2}+\frac{3}{4}\right)\log f. \tag{4.80}$$

Next we note that

$$s_1 = 1+\frac{1}{b\log f} > 1+\frac{1}{a\log f} \ge 1+\frac{x}{a\log f} = s \ge 1,$$

so that $1 \le s < s_1$, and hence, by Lemma 20, we obtain

$$\begin{aligned}\Re\left(\frac{1}{s-\rho}\right) &\le \frac{1}{1-(s_1-s)\Re(\frac{1}{s_1-\rho})}\Re\left(\frac{1}{s_1-\rho}\right)\\ &< \frac{1}{1-(s_1-s)(b/2+3/4)\log f}\Re\left(\frac{1}{s_1-\rho}\right),\end{aligned}$$

so that

$$\begin{aligned}\sum_{\substack{\rho \\ L_1(\rho)=0 \\ |\rho - s_0| \le c/2}} \Re\left(\frac{1}{s-\rho}\right) &< \frac{1}{1-(s_1-s)(b/2+3/4)\log f}\sum_{\substack{\rho \\ L_1(\rho)=0 \\ |\rho - s_0| \le c/2}} \Re\left(\frac{1}{s_1-\rho}\right)\\ &< \frac{(b+3/2)\log f}{1-(1/b-x/a)(b/2+3/4)}\\ &= \frac{(b+3/2)\log f}{(1-(1/b)(b/2+3/4))+(x/a)(b/2+3/4)}\\ &= \frac{(b+3/2)a\log f}{a(1-1/2-3/4b)+x(b/2+3/4)},\end{aligned}$$

that is,

$$\sum_{\substack{\rho \\ L_1(\rho)=0 \\ |\rho - s_0| \le c/2}} \Re\left(\frac{1}{s-\rho}\right) < \frac{(b+3/2)a\log f}{a(1/2-3/4b)+x(b/2+3/4)}. \tag{4.81}$$

Hence, by Lemmas 21 and 23, and (4.81), we have

$$\frac{L_1'(s)}{L_1(s)} \le \frac{(b+3/2)a\log f}{a(1/2-3/4b)+x(b/2+3/4)} + \frac{4a(\log A)(ac\log A+2)}{(ac\log A-2)^2}C\log f,$$

which is the required inequality.

Lemma 27 (see [17, p. 494]).

$$\int_1^{s_0} \frac{L_1'(s)}{L_1(s)}\,ds < 2\log\left(1+\frac{(9+6\sqrt{2})}{2a}\right) + 4(\log A)\frac{(ac\log A+2)C}{(ac\log A-2)^2}. \tag{4.82}$$

Proof. By Lemma 26 we obtain

$$\int_1^{s_0} \frac{L_1'(s)}{L_1(s)}\,ds < \int_0^1 \left(\frac{(b+3/2)a\log f}{a(1/2-3/4b)+x(b/2+3/4)} + \frac{4a(\log A)(ac\log A+2)C\log f}{(ac\log A-2)^2}\right)\frac{dx}{a\log f}$$

$$= 2\int_0^1 \frac{(b/2+3/4)\,dx}{a(1/2-3/4b)+x(b/2+3/4)} + \frac{4(\log A)(ac\log A+2)C}{(ac\log A-2)^2}$$

$$= 2\left[\log\left(a\left(\frac{1}{2}-\frac{3}{4b}\right)+x\left(\frac{b}{2}+\frac{3}{4}\right)\right)\right]_0^1 + \frac{4(\log A)(ac\log A+2)C}{(ac\log A-2)^2}$$

$$= 2\log\left(1+\frac{(b/2+3/4)}{a(1/2-3/4b)}\right) + \frac{4(\log A)(ac\log A+2)C}{(ac\log A-2)^2}$$

$$= 2\log\left(1+\frac{(9+6\sqrt{2})}{2a}\right) + \frac{4(\log A)(ac\log A+2)C}{(ac\log A-2)^2}$$

completing the proof of (4.82).

Lemma 28 (see [17, p. 495]). $-\log L_1(1) < 2\log\log f + \log D - \log(2\pi^2)$.

Proof. By Lemmas 10, 17 and 27, we obtain

$$-\log L_1(1) < \log a + \log\log f + \frac{1}{a\log f} + \log L_2(s_0) + \int_1^{s_0}\frac{L_1'(s)}{L_1(s)}\,ds$$

$$< \log a + \log\log f + \frac{1}{a\log A} + \log\log f + 2\log\left(1+\frac{(9+6\sqrt{2})}{2a}\right) + \frac{4(\log A)(ac\log A+2)C}{(ac\log A-2)^2}$$

$$= 2\log\log f + \log D - \log(2\pi^2) \quad \text{(by (1.8))}$$

completing the proof of Lemma 28.

Proposition 2. $h^*(K) > f/D(\log f)^2$.

Proof. From (2.14) and Lemma 28, we have

$$\log h^*(K) = \log f + \log L_1(1) - \log(2\pi^2) > \log f - 2\log\log f - \log D = \log\left(\frac{f}{D(\log f)^2}\right),$$

so that $h^*(K) > f/D(\log f)^2$, as required.

TABLE 1. Types of field K

Type	f	D	A	B	C	defining relation for B (> 0) and C (> 0)	number of fields
I	$8p$, $p \equiv 5 \pmod 8$	p	1	$1 \pmod 2$	$2 \pmod 4$	$p = B^2 + C^2$	1
II	$16p$, $p \equiv 3 \pmod 8$	2	p	1	1		1
III_1	$16p$, $p \equiv 5 \pmod 8$	2	p	1	1		1
III_2		$2p$	1	$1 \pmod 2$	$1 \pmod 2$	$2p = B^2 + C^2$	2
IV	pq, $p \equiv 3 \pmod 4$, $q \equiv 1 \pmod 8$, $(p/q) = -1$	q	p	$0 \pmod 4$	$1 \pmod 2$	$q = B^2 + C^2$	1
V_1	pq, $p \equiv 1 \pmod 8$, $q \equiv 5 \pmod 8$, $(p/q) = -1$	q	p	$2 \pmod 4$	$1 \pmod 2$	$q = B^2 + C^2$	1
V_2		pq	1	$2 \pmod 4$	$1 \pmod 2$	$pq = B^2 + C^2$	2
VI_1	pq, $p \equiv q \equiv 5 \pmod 8$, $(p/q) = -1$	p	q	$2 \pmod 4$	$1 \pmod 2$	$p = B^2 + C^2$	1
VI_2		q	p	$2 \pmod 4$	$1 \pmod 2$	$q = B^2 + C^2$	1

TABLE 2. Generators of G

I	$l_1 \equiv -1 \pmod 8$ $l_1 \equiv 1 \pmod p$ $\mathrm{ord}_G l_1 = 2$	$l_2 \equiv 5 \pmod 8$ $l_2 \equiv 1 \pmod p$ $\mathrm{ord}_G l_2 = 2$	$l_3 \equiv 1 \pmod 4$ $l_3 \equiv g^{(p-1)/4} \pmod p$ $\mathrm{ord}_G l_3 = 4$	$l_4 \equiv 1 \pmod 8$ $l_4 \equiv g^4 \pmod p$ $\mathrm{ord}_G l_4 = (p-1)/4$
II	$l_1 \equiv -1 \pmod{16}$ $l_1 \equiv 1 \pmod p$ $\mathrm{ord}_G l_1 = 2$	$l_2 \equiv 5 \pmod{16}$ $l_2 \equiv 1 \pmod p$ $\mathrm{ord}_G l_2 = 4$	$l_3 \equiv 1 \pmod 6$ $l_3 \equiv -1 \pmod p$ $\mathrm{ord}_G l_3 = 2$	$l_4 \equiv 1 \pmod{16}$ $l_4 \equiv g^2 \pmod p$ $\mathrm{ord}_G l_4 = (p-1)/2$
III	$l_1 \equiv -1 \pmod{16}$ $l_1 \equiv 1 \pmod p$ $\mathrm{ord}_G l_1 = 2$	$l_2 \equiv 5 \pmod{16}$ $l_2 \equiv 1 \pmod p$ $\mathrm{ord}_G l_2 = 4$	$l_3 \equiv 1 \pmod 6$ $l_3 \equiv g^{(p-1)/4} \pmod p$ $\mathrm{ord}_G l_3 = 4$	$l_4 \equiv 1 \pmod{16}$ $l_4 \equiv g^4 \pmod p$ $\mathrm{ord}_G l_4 = (p-1)/4$
IV	$l_1 \equiv -1 \pmod p$ $l_1 \equiv 1 \pmod q$ $\mathrm{ord}_G l_1 = 2$	$l_2 \equiv 1 \pmod p$ $l_2 \equiv h^{(q-1)/2^s} \pmod q$ $\mathrm{ord}_G l_2 = 2^s$	$l_3 \equiv g^2 \pmod p$ $l_3 \equiv 1 \pmod q$ $\mathrm{ord}_G l_3 = (p-1)/2$	$l_4 \equiv 1 \pmod p$ $l_4 \equiv h^{2^s} \pmod q$ $\mathrm{ord}_G l_4 = (q-1)/2^s$
V	$l_1 \equiv g^{(p-1)/2^r} \pmod p$ $l_1 \equiv 1 \pmod q$ $\mathrm{ord}_G l_1 = 2$	$l_2 \equiv 1 \pmod p$ $l_2 \equiv h^{(q-1)/4} \pmod q$ $\mathrm{ord}_G l_2 = 4$	$l_3 \equiv g^{2^r} \pmod p$ $l_3 \equiv 1 \pmod q$ $\mathrm{ord}_G l_3 = (p-1)/2^r$	$l_4 \equiv 1 \pmod p$ $l_4 \equiv h^4 \pmod q$ $\mathrm{ord}_G l_4 = (q-1)/4$
VI	$l_1 \equiv g^{(p-1)/4} \pmod p$ $l_1 \equiv 1 \pmod q$ $\mathrm{ord}_G l_1 = 4$	$l_2 \equiv 1 \pmod p$ $l_2 \equiv h^{(q-1)/4} \pmod q$ $\mathrm{ord}_G l_2 = 4$	$l_3 \equiv g^4 \pmod p$ $l_3 \equiv 1 \pmod q$ $\mathrm{ord}_G l_3 = (p-1)/4$	$l_4 \equiv 1 \pmod p$ $l_4 \equiv h^4 \pmod q$ $\mathrm{ord}_G l_4 = (q-1)/4$

5. Method of calculation of $h^*(K)$

We now describe how the computation of the relative class number $h^*(K)$ was carried out for all imaginary cyclic quartic fields K with conductor $f < 416,000$. First, a data file was created containing all those integers f in the range $1 < f < 416,000$, which are of one the forms specified in Proposition 1, together with the values of D, A, B, C for which the field $K = Q(\sqrt{A(D+B\sqrt{D})})$ has conductor f. Table 1 indicates the possibilities that can occur (p and q denote distinct odd primes).

In all there are 28,186 such 5-triples $(f, D, -A, B, C)$ with $f < 416,000$: 1338 of type I, 718 of type II, 2166 of type III, 10948 of type IV, 7764 of type V, and 5252 of type VI.

The generators of the group $G \cong \mathrm{Gal}(Q(e^{2\pi i/f})/Q)$ are the integers $l_1, l_2, l_3, l_4 \pmod f$ defined as in Table 2 (see for example [3, Chapter 4]). We note that g (resp. h) denotes a primitive root $(\bmod p)$ (resp. $(\bmod q)$) and 2^r (resp. 2^s) is the highest power of 2 dividing $p-1$ (resp. $q-1$). For $k = 1, 2, 3, 4$ we let $j_k = 0, 1, 2, 3$ denote the integer such that

$$l_k \in \alpha^{j_k} H. \tag{5.1}$$

In Table 3 we give the values of the j_k's corresponding to generators l_k of the 2-part of G. These values will help us in determining the generators of the 2-part of H.

Table 3. Values of j_k

I	$j_1 = 0$ or 2 $j_2 = 0$ or 2 $j_3 = 1$ or 3	IV	$j_1 = 2$ $j_2 = 1$ or 3
II	$j_1 = 0$ or 2 $j_2 = 1$ or 3 $j_3 = 0$ or 2	V_1	$j_1 = 2$ $j_2 = 1$ or 3
		V_2	$j_1 = 1$ or 3 $j_2 = 1$ or 3
III_1	$j_1 = 0$ or 2 $j_2 = 1$ or 3 $j_3 = 2$	VI_1	$j_1 = 1$ or 3 $j_2 = 2$
III_2	$j_1 = 0$ or 2 $j_2 = 1$ or 3 $j_3 = 1$ or 3	VI_2	$j_1 = 2$ $j_2 = 1$ or 3

Proof of values given in Table 3. We first note that $l \in \alpha^j H$ and

$$\mathrm{ord}_G l = 2 \Rightarrow 1 \in \alpha^{2j} H \Rightarrow 2j \equiv 0 \pmod 4 \Rightarrow j = 0 \text{ or } 2.$$

This establishes that $j_1 = 0$ or 2 and $j_2 = 0$ or 2 in I; $j_1 = 0$ or 2 and $j_3 = 0$ or 2 in II; and $j_1 = 0$ or 2 in III. We treat the remaining values case by case.

I. Let r be a prime $\equiv l_3 \pmod f$ so that (from Table 2)

$$\begin{cases} r \equiv 1 \pmod 8, \\ r \equiv g^{(p-1)/4} \pmod p. \end{cases}$$

Then, appealing to Table 1 and the law of quadratic reciprocity we have,

$$\left(\frac{D}{r}\right) = \left(\frac{p}{r}\right) = \left(\frac{r}{p}\right) = \left(\frac{g^{(p-1)/4}}{p}\right) = \left(\frac{g}{p}\right)^{(p-1)/4} = (-1)^{(p-1)/4} = -1,$$

and so $l_3 \in \alpha H$ or $\alpha^3 H$, that is $j_2 = 1$ or 3.

II. Let r be a prime $\equiv l_2 \pmod f$ so that (from Table 2)

$$\begin{cases} r \equiv 5 \pmod{16} \\ r \equiv 1 \pmod p. \end{cases}$$

Then, appealing to Table 1, we have $(D/r) = (2/r) = -1$, and so $l_2 \in \alpha H$ or $\alpha^3 H$, that is $j_2 = 1$ or 3.

III_1. Let r be a prime $\equiv l_2 \pmod f$ so that (from Table 2)

$$\begin{cases} r \equiv 5 \pmod{16}, \\ r \equiv 1 \pmod p. \end{cases}$$

Then, appealing to Table 1, we have $(D/r) = (2/r) = -1$, and so $l_2 \in \alpha H$ or $\alpha^3 H$, that is, $j_2 = 1$ or 3.

Now let r be a prime $\equiv l_3 \pmod f$ so that (from Table 2)

$$\begin{cases} r \equiv 1 \pmod{16}, \\ r \equiv g^{(p-1)/4} \pmod p. \end{cases}$$

Then, from Table 1, we have $(D/r) = (2/r) = 1$, and so $l_3 \in H$ or $\alpha^2 H$, that is, $j_3 = 0$ or 2. Now, by [19, Theorem, p. 257], we have

$$\left(\frac{A(D + B\sqrt{D})}{r}\right) = \left(\frac{-p(2+\sqrt{2})}{r}\right) = -\left(\frac{p}{r}\right)(-1)^{(r-1)/8}$$
$$= \left(\frac{p}{r}\right) = \left(\frac{r}{p}\right) = \left(\frac{g}{p}\right)^{(p-1)/4} = -1,$$

so that $l_3 \in \alpha^2 H$, that is $j_3 = 2$.

III_2. Similar to III_1.

IV. Let r be a prime $\equiv l_1 \pmod f$ so that (from Table 2)

$$\begin{cases} r \equiv -1 \pmod p, \\ r \equiv 1 \pmod q. \end{cases}$$

Then, as $D = q$ from Table 1, we have

$$\left(\frac{D}{r}\right) = \left(\frac{q}{r}\right) = \left(\frac{r}{q}\right) = \left(\frac{1}{q}\right) = 1,$$

and so $j_1 = 0$ or 2. Further, appealing to [19, Theorem p. 257], we have

$$\left(\frac{A(D+B\sqrt{D})}{r}\right) = \left(\frac{-p(q+B\sqrt{q})}{r}\right) = \left(\frac{-p}{r}\right)(-1)^{(r-1)(q-1)/8}\left(\frac{2}{r}\right)^B\left(\frac{r}{q}\right)_4$$
$$= \left(\frac{r}{p}\right) = \left(\frac{-1}{p}\right) = -1,$$

and so $j_1 = 2$.

Now let r be a prime $\equiv l_2 \pmod f$ so that (from Table 2)

$$\begin{cases} r \equiv 1 \pmod p, \\ r \equiv h^{(q-1)/2^s} \pmod q. \end{cases}$$

Then, as $D = q$ from Table 1, we have

$$\left(\frac{D}{r}\right) = \left(\frac{q}{r}\right) = \left(\frac{r}{q}\right) = \left(\frac{h}{q}\right)^{(q-1)/2^s} = -1,$$

and so $j_2 = 1$ or 3.

V_1. Let r be a prime $\equiv l_1 \pmod f$ so that (from Table 2)

$$\begin{cases} r \equiv g^{(p-1)/2^r} \pmod p, \\ r \equiv 1 \pmod q. \end{cases}$$

Then, as $D = q$ from Table 1, we have

$$\left(\frac{D}{r}\right) = \left(\frac{q}{r}\right) = \left(\frac{r}{q}\right) = 1,$$

and so $j_1 = 0$ or 2. Now by [19, Theorem, p. 257], we have

$$\left(\frac{A(D+B\sqrt{D})}{r}\right) = \left(\frac{-p(q+B\sqrt{q})}{r}\right) = \left(\frac{-p}{r}\right)(-1)^{(r-1)(q-1)/8}\left(\frac{2}{r}\right)^B\left(\frac{r}{q}\right)_4$$
$$= \left(\frac{-p}{r}\right)(-1)^{(r-1)/2} = \left(\frac{p}{r}\right) = \left(\frac{r}{p}\right) = -1,$$

so that $j_1 = 2$.

Now let r be a prime $\equiv l_2 \pmod f$ so that (from Table 2)

$$\begin{cases} r \equiv 1 \pmod p, \\ r \equiv h^{(q-1)/4} \pmod q. \end{cases}$$

Then, as $D = q$ from Table 1, we have

$$\left(\frac{D}{r}\right) = \left(\frac{q}{r}\right) = \left(\frac{r}{q}\right) = \left(\frac{h}{q}\right)^{(q-1)/4} = -1,$$

and so $j_2 = 1$ or 3.

V_2. Similar to V_1.

VI_1. Let r be a prime $\equiv l_1 \pmod f$ so (from Table 2)

$$\begin{cases} r \equiv g^{(p-1)/4} \pmod p, \\ r \equiv 1 \pmod q. \end{cases}$$

Then, as $D = p$ from Table 1, we have

$$\left(\frac{D}{r}\right) = \left(\frac{p}{r}\right) = \left(\frac{r}{p}\right) = \left(\frac{g}{p}\right)^{(p-1)/4} = -1,$$

so that $j_1 = 1$ or 3.

Now let r be a prime $\equiv l_3 \pmod f$ so (from Table 2)

$$\begin{cases} r \equiv 1 \pmod p, \\ r \equiv h^{(g-1)/4} \pmod q. \end{cases}$$

Then, as $D = p$ from Table 1, we have

$$\left(\frac{D}{r}\right) = \left(\frac{p}{r}\right) = \left(\frac{r}{p}\right) = 1,$$

and so $j_2 = 0$ or 2. Appealing to [19, Theorem, p. 257] we have

$$\left(\frac{A(D+B\sqrt{D})}{r}\right) = \left(\frac{-q(p+B\sqrt{p})}{r}\right) = \left(\frac{-q}{r}\right)(-1)^{(p-1)(r-1)/8}\left(\frac{2}{r}\right)^B\left(\frac{r}{p}\right)_4$$
$$= \left(\frac{-q}{r}\right)(-1)^{(r-1)/2} = \left(\frac{q}{r}\right) = \left(\frac{r}{q}\right) = \left(\frac{h}{q}\right)^{(q-1)/4} = -1,$$

so that $j_2 = 2$.

VI_2. Similar to VI_1.

Next we show that the generators of H can be taken as listed in Table 4 below. We remark that as the odd part of G is the same as the odd part of H, it suffices to prove that the 2-part of H is generated by the elements of even order listed in Table 4. The order of each generator in the table is given in parentheses.

TABLE 4. Generators of H

I	$l_1 l_3^{j_1}[2], l_2 l_3^{j_2}[2], l_4[(p-1)/4]$
II	$l_1 l_2^{j_1}[2], l_2^{j_3} l_3[2], l_4[(p-1)/4]$
III	$l_1 l_2^{j_1}[2], l_2^{3j_2j_3} l_3[4], l_4[(p-1)/4]$
IV	$l_1 l_2^2[2^{s-1}], l_3[(p-1)/2], l_4[(q-1)/2^s]$
V_1	$l_1 l_2^s[2^s], l_3[(p-1)/2^s], l_4[(q-1)/4]$
V_2	$l_1 l_2^{3j_1j_2}[2^r], l_3[(p-1)/2^r], l_4[(q-1)/4]$
VI_1	$l_1^2 l_1[4], l_3[(p-1)/4], l_4[(q-1)/4]$
VI_2	$l_1 l_2^2[4], l_3[(p-1)/4], l_4[(q-1)/4]$

Proof that the values given in Table 4 *are generators of* H.

I. A typical element of the 2-part of G is of the form

$$l_1^u l_2^v l_3^w \quad (u = 0, 1; v = 0, 1; w = 0, 1, 2, 3). \tag{5.2}$$

This element belongs to the 2-part of H if and only if

$$j_1 u + j_2 v + j_3 w \equiv 0 \pmod 4. \tag{5.3}$$

As j_1 and j_2 are even and j_3 is odd (Table 3), (5.3) is equivalent to $w \equiv -j_1 u - j_2 v \pmod 4$. Hence (5.2) becomes

$$l_1^u l_2^v l_3^{-j_1 u - j_2 v} = (l_1 l_3^{-j_1})^u (l_2 l_3^{-j_2})^v = (l_1 l_3^{j_1})^{-u} (l_2 l_3^{j_2})^v ,$$

showing that the 2-part of H is generated by $l_1 l_3^{j_1}$ and $l_2 l_3^{j_2}$.

II. A typical element of the 2-part of G is of the form

$$l_1^u l_2^v l_3^w \qquad (u = 0, 1; v = 0, 1, 2, 3; w = 0, 1). \tag{5.4}$$

This element belongs to the 2-part of H if and only if

$$j_1 u + j_2 v + j_3 w \equiv 0 \pmod 4. \tag{5.5}$$

As j_1 and j_3 are even and j_2 is odd (Table 3), (5.5) is equivalent to $v \equiv -j_1 v - j_3 w \pmod 4$. Hence (5.4) becomes

$$l_1^u l_2^{-j_1 u - j_3 w} l_3^w = (l_1 l_2^{-j_1})^u (l_2^{-j_3} l_3)^w = (l_1 l_2^{j_1})^{-u} (l_2^{j_3} l_3)^{-w}$$

showing that the 2-part of H is generated by $l_1 l_2^{j_1}$ and $l_2^{j_3} l_3$.

III. A typical element of the 2-part of G is of the form

$$l_1^u l_2^v l_3^w \qquad (u = 0, 1; v = 0, 1, 2, 3; w = 0, 1, 2, 3). \tag{5.6}$$

This element belongs to the 2-part of H if and only if

$$j_1 u + j_2 v + j_3 w \equiv 0 \pmod 4. \tag{5.7}$$

As j_1 is even and j_2 is odd (Table 3), (5.7) is equivalent to $v \equiv -j_1 u - j_2 j_3 w \pmod 4$. Hence (5.6) becomes

$$l_1^u l_2^{-j_1 u - j_2 j_3 w} l_3^w = (l_1 l_2^{-j_1})^u (l_2^{-j_2 j_3} l_3)^w = (l_1 l_2^{j_1})^{-u} (l_2^{3 j_2 j_3} l_3)^w$$

showing that the 2-part of H is generated by $l_1 l_2^{j_1}$ and $l_2^{3 j_2 j_3} l_3$.

IV. A typical element of the 2-part of G is of the form

$$l_1^u l_2^v \qquad (u = 0, 1; v = 0, 1, \ldots, 2^s - 1). \tag{5.8}$$

This element belongs to the 2-part of H if and only if

$$j_1 u + j_2 v \equiv 0 \pmod 4. \tag{5.9}$$

As $j_1 = 2$ and j_2 is odd, (5.9) is equivalent to $v \equiv 2u \pmod 4$ or

$$v \equiv 0 \pmod 2, \qquad u \equiv v/2 \pmod 2.$$

Hence (5.8) becomes

$$\begin{aligned} &l_1^{v/2} l_2^v \qquad (v = 0, 1, \ldots, 2^s - 1; v \text{ even}) \\ &= (l_1 l_2^2)^w \qquad (w = 0, 1, \ldots, 2^{s-1} - 1), \end{aligned}$$

showing that the 2-part of H is generated by $l_1 l_2^2$.

V. A typical element of the 2-part of G is of the form

$$l_1^u l_2^v \qquad (u = 0, 1, \ldots, 2^r - 1; v = 0, 1, 2, 3). \tag{5.10}$$

This element belongs to the 2-part of H if and only if

(5.11) $$j_1u+j_2v\equiv 0 \pmod 4.$$

V_1. As $j_1=2$ and j_2 is odd (Table 3), (5.11) is equivalent to $v\equiv 2u \pmod 4$. Hence (5.10) becomes $l_1^ul_2^{2u}=(l_1l_2^2)^u$, showing that the 2-part of H is generated by $l_1l_2^2$.

V_2. As j_1 and j_2 are both odd (Table 3), (5.11) is equivalent to $v\equiv 3j_1j_2u \pmod 4$. Hence (5.10) becomes

$$l_1^ul_2^{3j_1j_2u}=(l_1l_2^{3j_1j_2})^u,$$

showing that the 2-part of H is generated by $l_1l_2^{3j_1j_2}$.

VI. A typical element of the 2-part of G is of the form

(5.12) $$l_1^ul_2^v \qquad (u=0,1,2,3;v=0,1,2,3).$$

This element belongs to the 2-part of H if and only if

(5.13) $$j_1u+j_2v\equiv 0 \pmod 4.$$

VI_1. As j_1 is odd and $j_2=2$ (Table 3), (5.13) is equivalent to $u\equiv 2v \pmod 4$. Hence (5.12) becomes

$$l_1^{2v}l_2^v=(l_1^2l_2)^v,$$

showing that the 2-part of H is generated by $l_1^2l_2$.

VI_2. As $j_1=2$ and j_2 odd (Table 3), (5.13) is equivalent to $v\equiv 2u \pmod 4$. Hence (5.12) becomes

$$l_1^ul_2^{2u}=(l_1l_2^2)^u,$$

showing that the 2-part of H is generated by $l_1l_2^2$.

Finally the elements of $H,\alpha H,\alpha^2H,\alpha^3H$ were computed using the generators of H given in Table 4, and a count kept of those in each coset whose least positive residue $\pmod f$ was less than $f/2$. The relative class number $h^*(K)$ was then calculated from the formula (see (3.13))

(5.14) $$h^*(K)=((C_0-C_2)^2+(C_1-C_3)^2)/E,$$

where

(5.15) $$C_j=\sum_{\substack{0<n<f/2\\ n\in\alpha^jH}}1 \qquad (j=0,1,2,3)$$

and

(5.16) $$E=\begin{cases}8, & \text{if } f\equiv 0 \pmod 2,\\ 2, & \text{if } f\equiv 1 \pmod 2 \text{ and } 2\in H,\\ 18, & \text{if } f\equiv 1 \pmod 2 \text{ and } 2\in\alpha^2H,\\ 10, & \text{if } f\equiv 1 \pmod 2 \text{ and } 2\in\alpha H \text{ or } \alpha^3H.\end{cases}$$

The identities

$$C_0 + C_2 = \phi(f)/4, \qquad C_1 + C_3 = \phi(f)/4, \tag{5.17}$$

and the congruence

$$(C_0 - C_2)^2 + (C_1 - C_3)^2 \equiv 0 \pmod{E}, \tag{5.18}$$

served as checks on the calculation in order to reduce the chances of a computer error. Only the 10 fields K listed in §1 were found to have $h^*(K) = 2$ [8].

The program to compute the table of values of $h^*(K)$ was run at various times over a two-month period and the authors would like to express their appreciation to David Hutchinson, Associate Director (Services and Facilities), Computing and Communication Services, Carleton University, for allocating resources on Carleton University's Honeywell CP-6 systems in order to carry out these computations.

References

1. T. M. Apostol, *Introduction to analytic number theory*, Springer-Verlag, New York, 1976.
2. E. Artin, *The gamma function*, Holt, Rinehart and Winston, New York, 1964.
3. E. D. Bolker, *Elementary number theory*, Benjamin, New York, 1970.
4. E. Brown and C. J. Parry, *The 2-class group of certain biquadratic number fields*, J. Reine Angew. Math. **295** (1977), 61–71.
5. ——, *The 2-class group of biquadratic fields*. II, Pacific J. Math. **78** (1978), 11–26.
6. T. H. Gronwall, *Sur les séries de Dirichlet correspondant à des caractères complexes*, Rend. Circ. Mat. Palermo **35** (1913), 145–159.
7. K. Hardy, R. H. Hudson, D. Richman, K. S. Williams and N. M. Holtz, *Calculation of the class numbers of imaginary cyclic quartic fields*, Carleton-Ottawa Math. Lecture Notes Series, No. 7, July 1986, 201 pp.
8. K. Hardy, R. H. Hudson, D. Richman and K. S. Williams, *Table of the relative class numbers $h^*(K)$ of imaginary cyclic quartic fields K with $h^*(K) \equiv 2 \pmod 4$ and conductor $f < 416,000$*, Carleton-Ottawa Math. Lecture Note Series, No. 8, March 1987, 282 pp.
9. E. Landau, *Abschätzungen von Charaktersummen, Einheiten und Klassenzahlen*, Gött. Nachr., 1918, pp. 79–97.
10. ——, *Über die Wurzeln der Zetafunktion*, Math. Z. **20** (1924), 98–104.
11. ——, *Über Dirichletsche Reihen mit komplexer Charakteren*, J. Reine Angew. Math. **157** (1926), 26–32.
12. T. Metsänkylä, *Zero-free regions of Dirichlet's L-functions near the point* 1, Ann. Univ. Turku **139** (1970), 3–11.
13. G. Pólya, *Über die Verteilung der quadratischen Reste und Nichtreste*, Gött. Nachr., 1918, pp. 21–29.
14. B. Setzer, *The determination of all imaginary, quartic, abelian number fields with class number* 1, Math. Comp. **35** (1980), 1383–1386.
15. T. Tatuzawa, *On a theorem of Siegel*, Japan. J. Math. **21** (1951), 163–178.
16. K. Uchida, *Class numbers of imaginary abelian number fields*. II, Tôhoku Math. J. **23** (1971), 335–348.
17. ——, *Imaginary abelian number fields with class number one*, Tôhoku Math. J. **24** (1972), 487–499.
18. E. T. Copson, *An introduction to the theory of functions of a complex variable*, Oxford Univ. Press, 1960.

19. K. S. Williams, K. Hardy and C. Friesen, *On the evaluation of the Legendre symbol* $((A + B\sqrt{m})/p)$, Acta Arith. **45** (1985), 255–272.

20. M. Zitouni, *Quelques propriétés des corps cycliques de degré* 4, Séminaire DELANGE-PISOT-POITOU (Théorie des nombres), 14^e anné, 1972/73, no. 4, 8 pp.

Department of Mathematics and Statistics, Carleton University, Ottawa, Ontario, Canada K1S 5B6 (Current address of Kenneth Hardy and Kenneth S. Williams)

Department of Mathematics, University of South Carolina, Columbia, South Carolina 29208 (Current address of Richard H. Hudson and David Richman)