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CYCLIC QUARTIC FIELDS WITH RELATIVE INTEGRAL BASES OVER THEIR QUADRATIC SUBFIELDS

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ABSTRACT. Explicit conditions are given for a cyclic quartic field to have a relative integral basis over its unique quadratic subfield.

Throughout this paper, K denotes a cyclic quartic extension of the rational number field Q. By Theorem 1 of [**3**] we know that K can be expressed uniquely in the form

$$K = Q\left(\sqrt{A(D+B\sqrt{D})}\right), \tag{1}$$

where A, B, C, D are integers such that

$$\begin{aligned} &A \text{ is squarefree and odd,} \\ &D = B^2 + C^2 \text{ is squarefree}, B > 0,\ C > 0, \\ &(A, D) = 1. \end{aligned}$$

K possesses a unique quadratic subfield $k = Q(\sqrt{D})$. Although K possesses an integral basis over Q (an explicit integral basis is given in [**4**]) it may or may not have a relative integral basis (RIB) over k. In this paper we give a necessary and sufficient condition for K to have a RIB over k. This is done by using the integral basis for K over Q given in [**4**] to determine the relative discriminant $d(K/k)$ (see Lemma 2 below) and then appealing to the following theorem of Mann [**6**, Theorem 2].

THEOREM (MANN). *Let F be an algebraic number field and E a quadratic extension of F. Then E has a RIB over F if and only if $E = F(\sqrt{\Delta})$ for some $\Delta \in F$ with $d(E/F) = (\Delta)$.*

Our necessary and sufficient condition for K to have a RIB over k is given in terms of the fundamental unit ε (> 1) of $k = Q(\sqrt{D})$. Two cases naturally arise according as the norm $N_{k/Q}(\varepsilon) = +1$ or -1. First we prove

THEOREM 1. *If $N_{k/Q}(\varepsilon) = +1$ then K does not have a RIB over k.*

If $N_{k/Q}(\varepsilon) = -1$ we let (U, V) be the solution in positive integers of $U^2 - DV^2 = -1$ with V least. Setting $\varepsilon = (x + y\sqrt{D})/2$, where x and y are positive integers

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with $x \equiv y \pmod 2$, $x^2 - Dy^2 = -4$, we have

$$U + V\sqrt{D} = \begin{cases} \varepsilon, & \text{if } x \equiv y \equiv 0 \pmod 2, \\ \varepsilon^3, & \text{if } x \equiv y \equiv 1 \pmod 2. \end{cases}$$

We note that the case $x \equiv y \equiv 1 \pmod 2$ can only occur when $D \equiv 5 \pmod 8$. It is a classical result (see for example [**7**, Theorem 5.9]) that if $V > 1$ there is a unique pair of nonnegative coprime integers (S, T) such that

$$V = S^2 + T^2, \qquad T \equiv SU \pmod V. \tag{2}$$

If $V = 1$ we take $S = 1$, $T = 0$ so that (2) is satisfied in this case too. A familiar argument shows that S and T satisfy the congruence

$$(S^2 - T^2) + 2STU \equiv 0 \pmod{V^2},$$

so that we may define nonnegative integers M and N by

$$M = \frac{|U(S^2 - T^2) - 2ST|}{V^2}, \quad N = \frac{|(S^2 - T^2) + (2ST)U|}{V^2}.$$

As $M^2 + N^2 = D$, and D (> 1) is squarefree, M and N must be positive integers. Moreover we have

$$(U + V\sqrt{D})(X + Y\sqrt{D})^2 = (\pm M) + \sqrt{D}, \tag{3}$$

where

$$X = (T - SU)/V, \qquad Y = S.$$

We prove

THEOREM 2. *If $N_{k/Q}(\varepsilon) = -1$ then K has a RIB over k if and only if*

$$(B, C) = \begin{cases} (N, M), & \text{if } D \equiv 1 \pmod 4,\ B \equiv 1 \pmod 2, \\ (M, N), & \text{otherwise.} \end{cases}$$

It follows from Theorems 1 and 2 that if A and D are given integers such that

A is squarefree and odd,
D (> 1) is squarefree and representable as the sum of two squares,
$GCD(A, D) = 1$,

then either there are no pairs of positive integers (B, C) with $B^2 + C^2 = D$ for which $K = Q(\sqrt{A(D + B\sqrt{D})})$ has a RIB over k or there are one or two pairs according as $D \equiv 2 \pmod 8$ or $D \equiv 1 \pmod 4$.

EXAMPLE 1. $K = Q(\sqrt{5 + 2\sqrt{5}})$. Here $A = 1$, $B = 2$, $C = 1, D = 5$, $k = Q(\sqrt{5})$, $\varepsilon = (1 + \sqrt{5})/2$, $N_{k/Q}(\varepsilon) = -1$, $U = 2$, $V = 1$, $S = 1$, $T = 0$, $M = 2$, $N = 1$, and so, by Theorem 2, K has a RIB over k.

EXAMPLE 2. $K = Q(\sqrt{10 + \sqrt{10}})$. Here $A = 1$, $B = 1$, $C = 3$, $D = 10$, $k = Q(\sqrt{10})$, $\varepsilon = 3 + \sqrt{10}$, $N_{k/Q}(\varepsilon) = -1$, $U = 3$, $V = 1$, $S = 1$, $T = 0$, $M = 3$, $N = 1$, and so, by Theorem 2, K does *not* have a RIB over k.

In the case when K has a RIB over k we give an explicit relative integral basis for K/k.

THEOREM 3. *If K has a RIB over k then a RIB over K/k is given by*

$$\left\{1, \sqrt{A\sqrt{D}(U+V\sqrt{D})}\right\}, \quad \text{if } D \equiv 2 \pmod 8 \text{ or } D \equiv 1 \pmod 4, \; B \equiv 0 \pmod 2, \; A+B \equiv 3 \pmod 4;$$

$$\left\{1, \sqrt{2A\sqrt{D}(U+V\sqrt{D})}\right\}, \quad \text{if } D \equiv 1 \pmod 4, \; B \equiv 1 \pmod 2;$$

$$\left\{1, \tfrac{1}{2}\left(1+\sqrt{A\sqrt{D}(U+V\sqrt{D})}\right)\right\}, \quad \text{if } D \equiv 1 \pmod 4, \; B \equiv 0 \pmod 2, \; A+B \equiv 1 \pmod 4.$$

EXAMPLE 1 (CONT). Set $\alpha = \sqrt{5+2\sqrt{5}}$, $\beta = \sqrt{5-2\sqrt{5}}$, so that

$$\sqrt{5}\alpha = 2\alpha + \beta, \quad \sqrt{5}\beta = \alpha - 2\beta.$$

By Theorem 3 a RIB for $K = Q(\sqrt{5+2\sqrt{5}})$ over $k = Q(\sqrt{5})$ is given by $\{1, \alpha\}$. This is easily seen directly, as every integer of K is of the form (see [4])

$$\begin{aligned} x &+ y\left(\frac{1+\sqrt{5}}{2}\right) + z\left(\frac{\alpha+\beta}{2}\right) + w\left(\frac{\alpha-\beta}{2}\right) \\ &= \left(x + y\left(\frac{1+\sqrt{5}}{2}\right)\right)1 + \left((2w-z) + (z-w)\left(\frac{1+\sqrt{5}}{2}\right)\right)\alpha, \end{aligned}$$

where x, y, z, w are integers.

EXAMPLE 2 (CONT). We show directly that $K = Q(\sqrt{10+\sqrt{10}})$ does not possess a RIB over $k = Q(\sqrt{10})$. We set $\alpha = \sqrt{10+\sqrt{10}}$, $\beta = \sqrt{10-\sqrt{10}}$, so that

$$\sqrt{10}\alpha = \alpha + 3\beta, \quad \sqrt{10}\beta = 3\alpha - \beta.$$

The integers of K are of the form (see [4]) $x + y\sqrt{10} + z\alpha + w\beta$, where x, y, z, w are integers. Suppose that K has a relative integral basis over k. Such a basis may be taken in the form $\{1, \gamma\}$, where $\gamma = t\alpha + u\beta$ with integers t and u not both zero. Thus there must be integers a, b, c, d, e, f, g, h such that

$$\begin{aligned} \alpha &= (a + b\sqrt{10})1 + (c + d\sqrt{10})(t\alpha + u\beta), \\ \beta &= (e + f\sqrt{10})1 + (g + h\sqrt{10})(t\alpha + u\beta), \end{aligned}$$

and so we have

$$\begin{aligned} \alpha &= a + b\sqrt{10} + (tc + (t+3u)d)\alpha + (uc + (3t-u)d)\beta, \\ \beta &= e + f\sqrt{10} + (tg + (t+3u)h)\alpha + (ug + (3t-u)h)\beta. \end{aligned}$$

Equating coefficients of $1, \sqrt{10}, \alpha, \beta$, we obtain $a = b = e = f = 0$, and

$$\left\{ \begin{array}{l} tc + (t+3u)d = 1 \\ uc + (3t-u)d = 0 \end{array} \right\}, \qquad \left\{ \begin{array}{l} tg + (t+3u)h = 0 \\ ug + (3t-u)h = 1 \end{array} \right\}.$$

Solving for c, d and g, h, we obtain

$$\begin{aligned} c &= \frac{3t-u}{3t^2-2tu-3u^2}, \qquad d = \frac{-u}{3t^2-2tu-3u^2}, \\ g &= \frac{-t-3u}{3t^2-2tu-3u^2}, \qquad h = \frac{t}{3t^2-2tu-3u^2}. \end{aligned}$$

Note that $3t^2-2tu-3u^2 \neq 0$ as t and u are not both zero. As c, d, g, h are integers, we must have

$$3t^2-2tu-3u^2 \mid t, \qquad 3t^2-2tu-3u^2 \mid u.$$

Thus there are integers r and s such that

$$t=(3t^2-2tu-3u^2)r, \qquad u=(3t^2-2tu-3u^2)s,$$

and so

$$3t^2-2tu-3u^2=(3t^2-2tu-3u^2)^2(3r^2-2rs-3s^2),$$

giving

$$(3t^2-2tu-3u^2)(3r^2-2rs-3s^2)=1.$$

Hence we have

$$3t^2-2tu-3u^2=\pm 1,$$

and so

$$(3t-u)^2-10u^2=\pm 3,$$

which is impossible as $x^2 \equiv \pm 3 \pmod 5$ is insolvable.

We now begin the proofs of Theorems 1 and 2. We first calculate the relative different $\mathscr{D}(K/k)$. We set

$$\alpha=\sqrt{A(D+B\sqrt{D})}, \qquad \beta=\sqrt{A(D-B\sqrt{D})}.$$

LEMMA 1.

$$\mathscr{D}(K/k)=\begin{cases} 2(\alpha,\beta), & \text{if } B\equiv 1 \pmod 2, \\ (\alpha+\beta,\alpha-\beta), & \text{if } B\equiv 0 \pmod 2,\ A+B\equiv 3 \pmod 4, \\ \left(\dfrac{\alpha+\beta}{2},\dfrac{\alpha-\beta}{2}\right), & \text{if } B\equiv 0 \pmod 2,\ A+B\equiv 1 \pmod 4. \end{cases}$$

PROOF. We just give the details in the case $D\equiv 2 \pmod 8$ (so that $B\equiv C\equiv 1 \pmod 2$) as the other cases can be treated similarly. We will obtain $\mathscr{D}(K/k)$ from the relation

$$\mathscr{D}(K/k)\mathscr{D}(k/Q)=\mathscr{D}(K/Q). \tag{4}$$

We first calculate $\mathscr{D}(K/Q)$. An integral basis for K/Q in this case is given by (see [4]) $\{1,\sqrt{D},\alpha,\beta\}$. For convenience we set $\Omega_1=1, \Omega_2=\sqrt{D}, \Omega_3=\alpha, \Omega_4=\beta$, and define ideals X_1, X_2, X_3 of the ring O_K of integers of K by

$$X_j=(\Omega_1-\theta^j(\Omega_1),\quad \Omega_2-\theta^j(\Omega_2),\quad \Omega_3-\theta^j(\Omega_3),\quad \Omega_4-\theta^j(\Omega_4)),$$

where $\mathrm{Gal}(K/Q)=\langle\theta\rangle$, so that $\mathscr{D}(K/Q)=X_1X_2X_3$. As $\theta(\alpha)=\beta$, $\theta(\beta)=-\alpha$, $\theta(\sqrt{D})=-\sqrt{D}$, we have

$$X_1=X_3=(2\sqrt{D},\alpha-\beta,\alpha+\beta) \quad \text{and} \quad X_2=2(\alpha,\beta).$$

Next, making use of

$$\alpha^2=AD+AB\sqrt{D}, \quad \beta^2=AD-AB\sqrt{D}, \quad \alpha\beta=AC\sqrt{D},$$

we obtain

$$\begin{aligned} X_1X_3 &= (4D,(\alpha-\beta)^2,(\alpha+\beta)^2,2\sqrt{D}(\alpha-\beta),2\sqrt{D}(\alpha+\beta),\alpha^2-\beta^2) \\ &= 2\sqrt{D}I, \end{aligned}$$

where

$$I = (2\sqrt{D}, AC + A\sqrt{D}, AC - A\sqrt{D}, \alpha - \beta, \alpha + \beta, AB).$$

Now $2D \in I$, $AB \in I$, so as $(A, D) = 1$, $(B, D) = 1$, $A \equiv B \equiv 1 \pmod 2$, we have $(2D, AB) = (1)$, so that $I = (1)$, and $X_1 X_3 = (2\sqrt{D})$. Hence we have

$$\mathscr{D}(K/Q) = (2)^2(\sqrt{D})(\alpha, \beta). \tag{5}$$

Next we calculate $\mathscr{D}(k/Q)$. An integral basis for k in this case is $\{1, \sqrt{D}\}$ and, by the definition of the different, we have

$$\mathscr{D}(k/Q) = (1 - \theta(1), \sqrt{D} - \theta(\sqrt{D}))$$

so that

$$\mathscr{D}(k/Q) = (2\sqrt{D}). \tag{6}$$

Thus, from (4), (5), (6), we obtain

$$\mathscr{D}(K/k) = \frac{(2)^2(\sqrt{D})(\alpha, \beta)}{(2)(\sqrt{D})} = (2)(\alpha, \beta).$$

This completes the proof of Lemma 1 in this case.

Next we determine the relative discriminant $d(K/k)$.

LEMMA 2.

$$d(K/k) = \begin{cases} (2^3 A\sqrt{D}), & \textit{if } D \equiv 1 \pmod 4,\ B \equiv 1 \pmod 2, \\ (2^2 A\sqrt{D}), & \textit{if } D \equiv 2 \pmod 8, \ \textit{or} \\ & \quad D \equiv 1 \pmod 4,\ B \equiv 0 \pmod 2\ A + B \equiv 3 \pmod 4, \\ (A\sqrt{D}), & \textit{if } D \equiv 1 \pmod 4,\ B \equiv 0 \pmod 2,\ A + B \equiv 1 \pmod 4. \end{cases}$$

PROOF. We just give the details when $D \equiv 2 \pmod 8$, as the other cases can be treated similarly. We have (appealing to Lemma 1)

$$\begin{aligned} d(K/k) &= N_{K/k}(\mathscr{D}(K/k)) = (2)(\alpha, \beta)(2)(\beta, -\alpha) = (2)^2(\alpha, \beta)^2 \\ &= (2)^2(\alpha^2, \alpha\beta, \beta^2) = (2)^2(AD + AB\sqrt{D}, AC\sqrt{D}, AD - AB\sqrt{D}) \\ &= (2^2 A\sqrt{D})(\sqrt{D} + B, \sqrt{D} - B, C). \end{aligned}$$

Now, as $(2B, C) = 1$, we see that $(\sqrt{D} + B, \sqrt{D} - B, C) = (1)$, and so

$$d(K/k) = (2^2 A\sqrt{D})$$

as required.

PROOF OF THEOREM 1. Suppose $N_{k/Q}(\varepsilon) = +1$ and K has a RIB over k. Then, by Lemma 2 and Mann's theorem, there exists $\Delta \in O_k$ such that $K = Q(\sqrt{\Delta})$, and $(\Delta) = (2^j A\sqrt{D})$, where

$$j = \begin{cases} 3, & \text{if } D \equiv 1 \pmod 4,\ B \equiv 1 \pmod 2, \\ 2, & \text{if } D \equiv 2 \pmod 8, \ \text{or} \\ & \text{if } D \equiv 1 \pmod 4,\ B \equiv 0 \pmod 2,\ A + B \equiv 3 \pmod 4, \\ 0, & \text{if } D \equiv 1 \pmod 4,\ B \equiv 0 \pmod 2,\ A + B \equiv 1 \pmod 4. \end{cases}$$

Hence there is a unit $\eta \in O_k$ such that

$$\Delta = 2^j A\sqrt{D}\eta.$$

By Dirichlet's unit theorem we have

$$\eta = \pm\varepsilon^m, \qquad \text{for some integer } m,$$

and so

$$Q(\sqrt{A(D+B\sqrt{D})}) = Q(\sqrt{\pm 2^j A\sqrt{D}\varepsilon^m}).$$

Removing squares from under the radical sign on the right-hand side as appropriate and recalling that $Q(\sqrt{A(D+B\sqrt{D})})$ is a cyclic field, we see that

$$Q(\sqrt{A(D+B\sqrt{D})}) = Q(\sqrt{\pm 2^l A\sqrt{D}\varepsilon}),$$

where $l = 0, 1$. Moreover, as $Q(\sqrt{A(D+B\sqrt{D})})$ and $Q(\sqrt{\pm 2^l A\sqrt{D}\varepsilon})$ must both be totally real or both totally imaginary, we have

$$Q(\sqrt{A(D+B\sqrt{D})}) = Q(\sqrt{2^l A\sqrt{D}\varepsilon}).$$

Hence there exist $\alpha, \beta \in k$ such that

$$\sqrt{2^l A\sqrt{D}\varepsilon} = \alpha + \beta\sqrt{A(D+B\sqrt{D})}. \tag{7}$$

From (7) we see that

$$\sqrt{2^l A\sqrt{D}\varepsilon}\sqrt{A(D+B\sqrt{D})} = \frac{1}{2\beta}(2^l A\sqrt{D}\varepsilon + \beta^2 A(D+B\sqrt{D}) - \alpha^2) \in Q(\sqrt{D}).$$

Hence there exist rational numbers e and f such that

$$\sqrt{2^l A\sqrt{D}\varepsilon}\sqrt{A(D+B\sqrt{D})} = e + f\sqrt{D}. \tag{8}$$

Squaring (8) and taking norms, we obtain

$$2^{2l}A^2(-D)A^2DC^2 = (e^2 - Df^2)^2,$$

which is impossible. This completes the proof of Theorem 1.

LEMMA 3. *If $N_{k/Q}(\varepsilon) = -1$ then K has a RIB over k if and only if*

$$K = \begin{cases} Q(\sqrt{2A\sqrt{D}(U+V\sqrt{D})}), & \text{if } D \equiv 1 \pmod 4,\ B \equiv 1 \pmod 2, \\ Q(\sqrt{A\sqrt{D}(U+V\sqrt{D})}), & \text{otherwise.} \end{cases}$$

PROOF. We just treat the case $D \equiv 1 \pmod 4$, $B \equiv 1 \pmod 2$, as the other cases can be treated similarly. By Lemma 2 and Mann's theorem, K has a RIB over k if and only if

$$K = Q(\sqrt{2^3 A\sqrt{D}\lambda}), \tag{9}$$

for some positive unit λ in O_k. The unit λ must be positive for if λ were negative $Q(\sqrt{A(D+B\sqrt{D})})$ and $Q(\sqrt{2^3A\sqrt{D}\lambda})$ could not both be totally real or both totally imaginary. By Dirichlet's unit theorem, we have $\lambda = \varepsilon^m$ for some integer m. Recalling that $U + V\sqrt{D} = \varepsilon$ or ε^3 and removing squares from under the radical sign in (9) we see that K has a RIB over k if and only if

$$K = Q(\sqrt{2A\sqrt{D}(U+V\sqrt{D})^j}),$$

where $j = 0$ or 1. As K is cyclic we must have $j = 1$. This completes the proof of Lemma 3 in this case.

LEMMA 4. *If $N_{k/Q}(\varepsilon) = -1$, then we have*

$$Q(\sqrt{2A\sqrt{D}(U+V\sqrt{D})}) = Q(\sqrt{A(D+N\sqrt{D})}),$$
$$Q(\sqrt{A\sqrt{D}(U+V\sqrt{D})}) = Q(\sqrt{A(D+M\sqrt{D})}).$$

PROOF. This is clear from (3) and the fact

$$Q(\sqrt{2A\sqrt{D}(M+\sqrt{D})}) = Q(\sqrt{A\sqrt{D}(N+\sqrt{D})}).$$

PROOF OF THEOREM 2. Theorem 2 follows immediately from Lemmas 3 and 4 as the representation of K in the form (1) is unique.

PROOF OF THEOREM 3. In each case it is a simple matter to check that the given set of elements has discriminant equal to $d(K/k)$ (the value of which is given in Lemma 2). Appealing to Theorem 2 and Lemma 4, it is easy to check that in each case the elements lie in K. The only element which is not obviously an algebraic integer is

$$\gamma = \tfrac{1}{2}(1+\sqrt{A\sqrt{D}(U+V\sqrt{D})}).$$

Since γ satisfies

$$\gamma^2 - \gamma + \tfrac{1}{4}(1 - AVD - AU\sqrt{D}) = 0$$

it suffices to show that $\frac{1}{4}(1 - AVD - AU\sqrt{D})$ is an integer of k. Since $D \equiv 1 \pmod 4$ (in this case) and $U^2 - V^2D = -1$, we have $U \equiv 0 \pmod 2$ and $V \equiv 1 \pmod 2$. Moreover we have $V \equiv 1 \pmod 4$ as $U^2 \equiv -1 \pmod V$. Hence $1 - AVD$ and AU are both even, and so, it suffices to show that

$$1 - AVD \equiv -AU \pmod 4,$$

or equivalently

$$A(VD - U) \equiv 1 \pmod 4.$$

We consider two cases according as $U \equiv 0 \pmod 4$ or $U \equiv 2 \pmod 4$. If $U \equiv 0 \pmod 4$ then, from $U^2 - V^2D = -1$, we deduce that $V^2D \equiv 1 \pmod 8$, so that $D \equiv 1 \pmod 8$, and thus $B \equiv 0 \pmod 4$. Hence, as $A + B \equiv 1 \pmod 4$ (in this case), we obtain $A \equiv 1 \pmod 4$, giving $A(VD - U) \equiv 1 \pmod 4$. If $U \equiv 2 \pmod 4$ then as above we conclude $D \equiv 5 \pmod 8$, $B \equiv 2 \pmod 4$, $A \equiv 3 \pmod 4$ and $A(VD - U) \equiv 1 \pmod 4$. This completes the proof of Theorem 3.

We conclude by remarking that Xianke [**8**] has given a less explicit form of Theorems 1, 2, 3. Relative integral bases for bicyclic quartic fields over their quadratic subfields are considered in [**1**, **2** and **5**].

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