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NOTE ON THE KLOOSTERMAN SUM

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ABSTRACT. The Kloosterman sum

$$\sum_{x=0;\,(x,p)=1}^{p^\alpha-1} \exp(2\pi i n(x+\bar{x})/p^\alpha),$$

where p is an odd prime, $\alpha \geqq 2$ and $(n, p)=1$, is evaluated in a very short direct way.

Let p denote an odd prime, α a positive integer, and n an integer such that $(n, p)=1$. The Kloosterman sum $A_{p^\alpha}(n)$ is given by

$$A_{p^\alpha}(n) = \sum_{x=0}^{p^\alpha-1}{}' \exp(2\pi i n(x+\bar{x})/p^\alpha), \tag{1}$$

where the dash (′) indicates that x only takes values from $0, 1, \cdots, p^\alpha-1$ which are coprime with p, and $\bar{x}$ is the unique solution of the congruence $x\bar{x}\equiv 1 \pmod{p^\alpha}$ satisfying $0<\bar{x}<p^\alpha$. Salié [3] has evaluated $A_{p^\alpha}(n)$ explicitly when $\alpha\geqq 2$. His evaluation is based upon induction. A direct (but fairly long) proof has been given by Whiteman [4] which requires results concerning Ramanujan sums. In this note we give a modification of Salié's original argument which gives a very short direct evaluation of $A_{p^\alpha}(n)$. (The referee has kindly pointed out that essentially the same technique has been used by Estermann [2], Carlitz [1].)

We let $\gamma=\alpha-[\alpha/2]$ and $\delta=[\alpha/2]$, so that $\alpha=\gamma+\delta$, $2\gamma\geqq\alpha$ and $\gamma\geqq\delta\geqq 1$. Setting $x=u+vp^\gamma$ ($u=0, 1, \cdots, p^\gamma-1$; $v=0, 1, \cdots, p^\delta-1$) in (1), so that $\bar{x}\equiv\bar{u}-\bar{u}^2vp^\gamma \pmod{p^\alpha}$, we obtain

$$\begin{aligned} A_{p^\alpha}(n) &= \sum_{u=0}^{p^\gamma-1}{}' \sum_{v=0}^{p^\delta-1} \exp(2\pi i n((u+vp^\gamma)+(\bar{u}-\bar{u}^2vp^\gamma))/p^\alpha) \\ &= \sum_{u=0}^{p^\gamma-1}{}' \exp(2\pi i n(u+\bar{u})/p^\alpha) \sum_{v=0}^{p^\delta-1} \exp(2\pi i n v(1-\bar{u}^2)/p^\delta) \\ &= p^\delta \sum_{u=0;\,u^2\equiv 1 \pmod{p^\delta}}^{p^\gamma-1} \exp(2\pi i n(u+\bar{u})/p^\alpha). \end{aligned}$$

If α is even, say $\alpha=2\beta$, then $\gamma=\delta=\beta$, and as the solutions u of $u^2\equiv 1 \pmod{p^\beta}$ in the range $0\leqq u\leqq p^\beta-1$ are $u=1$, $p^\beta-1$ (so that

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$\bar{u}=1, p^{2\beta}-p^{\beta}-1$ respectively) we have

$$A_{p^{2\beta}}(n) = p^{\beta}\{\exp(4\pi in/p^{2\beta}) + \exp(-4\pi in/p^{2\beta})\} = 2p^{\beta}\cos(4\pi n/p^{2\beta}).$$

If α is odd, say $\alpha=2\beta+1$, then $\gamma=\beta+1$, $\delta=\beta$, and as the solutions u of $u^2\equiv 1 \pmod{p^{\beta}}$ in the range $0\leqq u\leqq p^{\beta+1}-1$ are $u=1+wp^{\beta}$ $(w=0, 1, \cdots, p-1)$, $-1+wp^{\beta}$ $(w=1, 2, \cdots, p)$ (so that $\bar{u}\equiv 1 -wp^{\beta}+w^2p^{2\beta}$, $-1-wp^{\beta}-w^2p^{2\beta} \pmod{p^{2\beta+1}}$ respectively) we have

$$\begin{aligned} A_{p^{2\beta+1}}(n) &= p^{\beta}\Big\{\exp(4\pi in/p^{2\beta+1})\sum_{w=0}^{p-1}\exp(2\pi inw^2/p) \\ &\qquad + \exp(-4\pi in/p^{2\beta+1})\sum_{w=1}^{p}\exp(-2\pi inw^2/p)\Big\} \\ &= 2(n/p)p^{\beta+1/2}\cos(4\pi n/p^{2\beta+1}), \quad \text{if } p\equiv 1 \pmod 4, \\ &= -2(n/p)p^{\beta+1/2}\sin(4\pi n/p^{2\beta+1}), \quad \text{if } p\equiv 3 \pmod 4, \end{aligned}$$

using the well-known result [4]

$$\sum_{w=0}^{p-1}\exp(2\pi inw^2/p) = (n/p)i^{(p-1)^2/4}p^{1/2}.$$

References

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