Delta 2 (1970), 1-11.

ASYMPTOTIC BEHAVIOUR OF THE n^{th} TERM OF CERTAIN SUBSEQUENCES OF THE NATURAL NUMBERS

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Let $A = \{a(1),\ a(2),\ \ldots,\ a(n),\ \ldots\}$ be an infinite subsequence of the natural numbers. Number theory provides us with a wealth of examples of such subsequences A for which the asymptotic behaviour is known (as $x \to +\infty$) of the number $\pi_A(x)$ of elements in A, which are less than or equal to the real number x. (For a few such examples see [2]—[12]). For example [3] if A is the subsequence of squarefree integers it is known that

$$\pi_A(x) = \frac{6}{\pi^2} x + O(x^{\frac{1}{2}}), \text{ as } x \to +\infty.$$

Hence as $\pi_A(a(n)) = n$, we have

$$n = \frac{6}{\pi^2} a(n) + O((a(n))^{\frac{1}{2}}),$$

from which we deduce

$$a(n) = O(n),$$

and so
$$n = \frac{6}{\pi^2} a(n) + O(n^{\frac{1}{2}}), \text{ that is,}$$

$$a(n) = \frac{\pi^2}{6} n + O(n^{\frac{1}{2}}), \text{ as } n \to +\infty.$$

Thus we have deduced the asymptotic behaviour of $a(n)$ from the known asymptotic behaviour of $\pi_A(x)$. It is the purpose of this paper to do this for a general subsequence A for which the asymptotic behaviour of $\pi_A(x)$ is known. We suppose that an asymptotic formula for $\pi_A(x)$ is known of the following type :

$$(1) \qquad \pi_A(x) = f(x) + O(g(x)), \quad \text{as} \quad x \to +\infty,$$

where the constant implied by the O-symbol is independent of x. It will always be understood that such an expression as (1) is a genuine asymptotic formula, that is, $f(x)$ is the "main term", so that $\pi_A(x) \sim f(x)$, as $x \to +\infty$, and $O(g(x))$ is the "error term". This is guaranteed by

$$\lim_{x \to +\infty} \frac{g(x)}{f(x)} = 0.$$

We prove

Theorem. Let $A = \{a(1), a(2), \ldots\}$ be an infinite subsequence of the natural numbers for which an asymptotic formula (1) is known, where for all sufficiently large x, $f'(x)$ and $g'(x)$ both exist, and satisfy

$$(2) \quad f'(x) > O, \quad g'(x) > O, \quad \frac{f(x)}{f'(x)} = O(x), \quad \frac{g'(x)}{g(x)} = O\left(\frac{1}{x}\right),$$

as $x \to +\infty$.

Then if $(k(x), h(x))$ is a pair of real-valued functions with $h(x) = o(x)$, as $x \to +\infty$, such that for all sufficiently large x we have

$$(3) \qquad f(k(x)) = x + O(h(x)),$$

then

$$(4) \qquad a(n) = k(n) + O\left(\frac{k(n)}{n} \max\,(g(k(n)), h(n))\right),$$

as $n \to +\infty$.

We note that (2) implies the existence of $f^{-1}(x)$ for all sufficiently large x and so there is always a pair $(k(x), h(x))$ satisfying (3), namely $(k(x), h(x)) = (f^{-1}(x), 0)$. With this choice the theorem gives

$$a(n) = f^{-1}(n) + O\left(\frac{f^{-1}(n)\, g(f^{-1}(n))}{n}\right), \text{ as } n \to \infty. \tag{5}$$

However as we shall see in the examples concluding this paper, it is often more convenient to apply (4) with $k(x) \neq f^{-1}(x)$ rather than (5).

In the proof of the theorem we make use of the following theorem due to Entringer [1], namely, if $r(x) \to +\infty$ and $r(x) \sim s(x)$ as $x \to +\infty$, and $t(x)$ is monotonic and

$$\frac{t'(x)}{t(x)} = O\left(\frac{1}{x}\right)$$

for all sufficiently large x, then $t(r(x)) \sim t(s(x))$, as $x \to +\infty$.

Proof of Theorem, As $f'(x) > 0$, for all sufficiently large x, $f^{-1}(x)$ exists and is differentiable with positive derivative $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$, for all sufficiently large x. Moreover as A is an infinite subsequence we have $\pi_A(x) \to +\infty$, as $x \to +\infty$. But $\pi_A(x) \sim f(x)$, as $x \to +\infty$, so we must have $f(x) \to +\infty$, as $x \to +\infty$. Thus $f^{-1}(x) \to +\infty$, as $x \to +\infty$, and so choosing $y = f^{-1}(x)$ in $\dfrac{f(y)}{f'(y)} = O(y)$, as $y \to +\infty$, we obtain

$$\frac{(f^{-1})'(x)}{f^{-1}(x)} = \frac{1}{xf^{-1}(x)} \cdot \frac{f(f^{-1}(x))}{f'(f^{-1}(x))} \tag{6}$$

$$= \frac{1}{xf^{-1}(x)} \cdot O(f^{-1}(x)) = O\left(\frac{1}{x}\right),$$

as $x \to +\infty$. Now from (3), as $h(x) = o(x)$, as $x \to +\infty$, we have

$$f(k(x)) \sim x, \quad \text{as } x \to +\infty. \tag{7}$$

Thus for all sufficiently large x, we have

(8) $$f(k(x)) \geqslant \frac{x}{2}.$$

From (6) and (7) by Entringer's theorem we have

(9) $k(x) = f^{-1}(f(k(x))) \sim f^{-1}(x)$, as $x \to +\infty$, and so for all sufficiently large x we have

(10) $$\frac{1}{2}k(x) \leqslant f^{-1}(x) \leqslant \frac{3}{2}k(x).$$

From (9) we deduce $k(x) \to +\infty$, as $x \to +\infty$, and so as $f(x)$ is monotonic increasing we have from (10) for all sufficiently large x

(11) $$x \leqslant f(\frac{3}{2}k(x)).$$

Hence from (8) and (11) we have for all sufficiently large x

(12) $$\max(x, f(k(x))) \leqslant f(\frac{3}{2}k(x)) \min(x, f(k(x))) \geqslant \frac{x}{2}.$$

Now by the mean value theorem there exists $c(x)$ satisfying $\min(x, f(k(x)) \leqslant c(x) \leqslant \max(x, f(k(x)))$ and such that

(13) $$|f^{-1}(x) - k(x)| = |f^{-1}(x) - f^{-1}(f(k(x)))|$$
$$= |(f^{-1})'(c(x))(x - f(k(x)))|.$$

From (12) we deduce that

(14) $$\frac{x}{2} \leqslant c(x) \leqslant f(\frac{3}{2}k(x)).$$

and so as $f^{-1}(x)$ is monotonic increasing for all sufficiently large x, we have

$$f^{-1}(c(x)) \leqslant \frac{3}{2}k(x).$$

Hence from (6) and (14) we have

$$(15) \quad (f^{-1})'(c(x)) = O\left(\frac{f^{-1}(c(x))}{c(x)}\right) = O\left(\frac{k(x)}{x}\right),$$

as $x \to +\infty$.

Thus from (3), (13) and (15) we deduce

$$(16) \quad | f^{-1}(x) - k(x) | = O\left(\frac{k(x)h(x)}{x}\right), \text{ as } x \to +\infty.$$

Now taking $x = a(n)$, $n \to +\infty$, in (1) we obtain

$$(17) \qquad n = \pi_A(a(n)) = f(a(n)) + O(g(a(n))),$$

as $n \to +\infty$, that is,

$$(18) \qquad n \sim f(a(n)), \text{ as } n \to +\infty,$$

and so in particular for all sufficiently large n we have

$$(19) \qquad f(a(n)) \geqslant \frac{n}{2}.$$

From (18) by Entringer's theorem we have

$$(20) \qquad f^{-1}(n) \sim a(n), \text{ as } n \to \infty,$$

and so in particular for all sufficiently large n we have

$$(21) \qquad \frac{1}{2}a(n) \leqslant f^{-1}(n) \leqslant \frac{3}{2}a(n).$$

Thus as $f(x)$ is increasing for all sufficiently large x, and $a(n) \to +\infty$, as $n \to +\infty$, we deduce from (21) that for all sufficiently large n,

$$(22) \qquad n \leqslant f\left(\frac{3}{2}a(n)\right).$$

Hence from (19) and (22) we have for all sufficiently large n

$$(23)\qquad \max\,(n, f(a(n))) \leqslant f\left(\frac{3}{2}a(n)\right),\qquad \min\,(n, f(a(n))) \geqslant \frac{n}{2}.$$

Now by the mean value theorem there exists $d(n)$ satisfying $\min\,(n, f(a(n))) \leqslant d(n) \leqslant \max\,(n, f(a(n)))$ such that

$$(24)\qquad |\,a(n)-f^{-1}(n)\,| = |\,f^{-1}(f(a(n)))-f^{-1}(n)\,| = |\,(f^{-1})'(d(n))(f(a(n))-n)\,|\,.$$

From (23) we deduce that for all sufficiently large n

$$(25)\qquad \frac{n}{2} \leqslant d(n) \leqslant f\left(\frac{3}{2}\,a(n)\right),$$

and so as $f^{-1}(x)$ is monotonic increasing for all sufficiently large x, we have

$$f^{-1}(d(n)) \leqslant \frac{3}{2}\,a(n).$$

Hence from (6) and (25) we have

$$(26)\qquad (f^{-1})'(d(n)) = O\left(\frac{f^{-1}(d(n))}{d(n)}\right) = O\left(\frac{a(n)}{n}\right),\text{ as } n \to +\infty.$$

Moreover from (9) and (20) we have

$$(27)\qquad a(n) \sim f^{-1}(n) \sim k(n),\text{ as } n \to +\infty,$$

so that (26) becomes

$$(28)\qquad (f^{-1})'(d(n)) = O\left(\frac{k(n)}{n}\right),\text{ as } n \to +\infty.$$

From (2), $g(x)$ satisfies the conditions of Entringer's theorem and so by (27) we can deduce

$$(29)\qquad g(a(n)) \sim g(k(n)),\quad\text{as } n \to +\infty,$$

so that from (17), (24), (28), (29) we obtain

$$(30) \qquad | a(n) - f^{-1}(n) | = O\left(\frac{k(n) g(k(n))}{n} \right), \text{ as } n \to +\infty.$$

(4) now follows from (16) and (30) in view of the inequality

$$| a(n) - k(n) | \leqslant | a(n) - f^{-1}(n) | + | f^{-1}(n) - k(n) | .$$

We remark that (4) is a genuine asymptotic formula as

$h(n) = o(n)$ and $g(k(n)) = o(f(k(n)) = o(n)$, as $n \to +\infty$.

We conclude this paper with two examples.

Example 1. Let $a(n) = n^{th}$ integer which is the sum of two squares. Then it is known [5] (p. 261) that

$$\pi_A(x) = \frac{Bx}{\log^{\frac{1}{2}} x} + O\left(\frac{x}{\log^{\frac{3}{4}} x} \right), \text{ as } x \to +\infty,$$

where
$$B = \frac{1}{\sqrt{2}} \prod_{r=1}^{\infty} \left(1 - \frac{1}{r^2} \right)^{-\frac{1}{2}}$$

Thus we may take

$$f(x) = \frac{Bx}{\log^{\frac{1}{2}} x} \text{ and } g(x) = \frac{x}{\log^{\frac{3}{4}} x} .$$

It is easily verified that the conditions given in (2) are satisfied. Further as (see below)

$$(31) \qquad f\left(\frac{x \log^{\frac{1}{2}} x}{B} \right) = x + o\left(\frac{x \log \log x}{\log x} \right), \text{ as } x \to +\infty.$$

we can choose

$$(k(x), h(x)) = \left(\frac{x \log^{\frac{1}{2}} x}{B}, \frac{x \log \log x}{\log x} \right).$$

Then by the theorem we have

$$a(n) = \frac{n \log^{\frac{1}{2}} n}{B} + O\,(n \log^{\frac{1}{4}} n), \text{ as } n \to +\infty.$$

Proof of (31). For $x \geqslant \exp(B^2)$,

so that
$$\frac{\log^{\frac{1}{2}} x}{B} \geqslant 1,$$

we have

$$x - f\left(\frac{x \log^{\frac{1}{2}} x}{B}\right) = x - x\left(\frac{\log x}{\log\left(\frac{1}{B}\, x \log^{\frac{1}{2}} x\right)}\right)^{\frac{1}{2}}$$

$$\geqslant x - x\left(\frac{\log x}{\log x}\right)^{\frac{1}{2}} = 0,$$

so that as $x \to +\infty$ we have

$$\left| x - f\left(\frac{x \log^{\frac{1}{2}} x}{B}\right)\right| = x - f\left(\frac{x \log^{\frac{1}{2}} x}{B}\right)$$

$$= x\, \frac{\left\{\log^{\frac{1}{2}}\left(\frac{1}{B}\, x \log^{\frac{1}{2}} x\right) - \log^{\frac{1}{2}} x\right\}}{\log^{\frac{1}{2}}\left(\frac{1}{B}\, x \log^{\frac{1}{2}} x\right)}$$

$$\leqslant \frac{x}{\log^{\frac{1}{2}} x}\left\{\log^{\frac{1}{2}}\left(\frac{1}{B}\, x \log^{\frac{1}{2}} x\right) - \log^{\frac{1}{2}} x\right\}$$

$$= x\left[\left(1 + \frac{\log\left(\frac{\log^{\frac{1}{2}} x}{B}\right)}{\log x}\right)^{\frac{1}{2}} - 1\right]$$

$$\leqslant \frac{x}{2} \frac{\log\left(\frac{\log^{\frac{1}{2}} x}{B}\right)}{\log x}$$

$$= O\left(\frac{x \log \log x}{\log x}\right)$$

as required.

Example 2. Let $a(n) = n^{th}$ prime number. It is well-known [5] (p. 250) that one form of the prime number theorem with error term is

$$\Pi_A(x) = li(x) + O(x\, e^{-c\sqrt{\log x}}), \text{ as } x \to +\infty,$$

where c is a positive constant and $li(x)$ is the logarithmic integral

$$\int_2^x \frac{dt}{\log t}.$$

Thus we may take $f(x) = li(x)$ and $g(x) = xe^{-c\sqrt{\log x}}$. It is easily verified that the conditions given in (2) are satisfied. Further as (see below).

$$(32) \qquad li(x \log x) = x + O\left(\frac{x \log \log x}{\log x}\right), \text{ as } x \to +\infty,$$

we can choose

$$(k(x), h(x)) = \left(x \log x, \frac{x \log \log x}{\log x}\right).$$

Then by the theorem we have

$$a(n) = n \log n + O(n \log \log n), \text{ as } n \to +\infty.$$

Proof of (32). We have on integrating by parts

$$li(x \log x) - x = \int_2^{x \log x} \frac{dt}{\log t} - x$$

$$= \frac{-x \log \log x}{\log (x \log x)} + \int_2^{x \log x} \frac{dt}{\log^2 t} + O(1)$$

$$= O\left(\frac{x \log \log x}{\log x}\right),$$

as $\int_2^{x \log x} \frac{dt}{\log^2 t} = \left(\int_2^{\sqrt{x}} + \int_{\sqrt{x}}^{x \log x}\right) \frac{dt}{\log^2 t}$

$$= O(\sqrt{x}) + O\left(\frac{x \log x - \sqrt{x}}{\log^2 \sqrt{x}}\right)$$

$$= O\left(\frac{x}{\log x}\right).$$

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