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A SUM OF FRACTIONAL PARTS—II

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Let L_n denote the set of points $\underset{\sim}{x}=(x_1, x_2, \ldots, x_n)$ with integral co-ordinates in Euclidean n-space, where $n \geqslant 3$. For any odd prime p, let $C=C(p, n)$ be the set of points of L_n in the cube $0 \leqslant x_i < p$ $(i=1, 2, \ldots, n)$. Suppose $f(\underset{\sim}{x})$ is any polynomial of degree $d \geqslant 1$ in $x_1, \ldots, x_n$ with integral coefficients, which does not vanish identically (mod p). I let $\{a\}$ denote the fractional part of the real number a. In [1], I considered the problem of estimating

$$(1) \qquad \sum_{\underset{\sim}{x} \in C} \left\{ \frac{f(\underset{\sim}{x})}{p} \right\},$$

for large primes p. I proved that

$$(2) \qquad \sum_{\underset{\sim}{x} \in C} \left\{ \frac{f(\underset{\sim}{x})}{p} \right\} = \frac{1}{2} p^n + O\left(p^{n-\frac{1}{2}} \log p\right),$$

as $p \to \infty$, where here (and throughout this paper) the constant implied in the O-symbol depends only upon n and d. I conjectured, however, that the better result

$$(3) \qquad \sum_{\underset{\sim}{x} \in C} \left\{ \frac{f(\underset{\sim}{x})}{p} \right\} = \frac{1}{2} p^n + O\left(p^{n-\frac{1}{2}}\right)$$

holds. It is the purpose of this paper to prove the following

Theorem. For *almost all homogeneous* polynomials $f(\underset{\sim}{x})$ of degree $d \geqslant 1$ in the $n \geqslant 3$ variables $\underset{\sim}{x}=(x_1, \ldots, x_n)$, we have

$$\sum_{\underset{\sim}{x} \in C} \left\{ \frac{f(\underset{\sim}{x})}{p} \right\} = \frac{1}{2} p^n + O\left(p^{n-\frac{1}{2}}\right),$$

as $p \to \infty$.

We begin by introducing a little notation. We write

$$(4) \qquad f(\underset{\sim}{x}) = \sum_{\substack{i_1, \ldots, i_n = 0 \\ i_1 + \ldots + i_n = d}}^{d} a_{i_1 \ldots i_n} x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n},$$

where, without loss of generality, we can take

(5) $0 \leqslant a_{i_1 \cdots i_n} \leqslant p-1$.

As $f(\underset{\sim}{x})$ is assumed not to vanish identically (mod p), not all the $a_{i_1 \cdots i_n}$ will vanish. In all, there will be

(6) $$k \equiv k(n, d) = \binom{n+d-1}{d}$$

coefficients $a_{i_1 \cdots i_n}$. We note that

(7) $$k > n.$$

Lastly, we let $e(t)$ denote $\exp(2\pi i t p^{-1})$ for all real t. We shall need the following lemmas.

Lemma 1. If $t \not\equiv 0 \pmod p$

(8) $$\sum_{\deg f = d} e(t f(\underset{\sim}{x})) = \begin{cases} p^k - 1, & \text{if } \underline{x} \equiv \underline{0} \\ -1, & \text{if } \underline{x} \not\equiv \underline{0} \end{cases}.$$

Proof. We have

$$\sum_{\deg f = d} e(t f(\underset{\sim}{x})) \qquad +1$$

(9)

$$= \prod_{\substack{i_1, \ldots, i_n = 0 \\ i_1 + \ldots + i_n = d}}^{d} \left\{ \sum_{a_{i_1 \cdots i_n} = 0}^{p-1} e(t\, a_{i_1 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n}) \right\}.$$

If $\underline{x} \equiv \underline{0}$, the right hand side of (9) is just

$$\prod_{\substack{i_1, \ldots, i_n = 0 \\ i_1 + \ldots + i_n = d}}^{d} \sum_{a_{i_1 \cdots i_n} = 0}^{p-1} 1 = p^k.$$

If $\underline{x} \not\equiv \underline{0}$, there is an integer l $(1 \leqslant l \leqslant n)$ such that $x_l \not\equiv 0$, so

$$\sum_{a_{0 \ldots 0\, l\, 0 \ldots 0} = 0}^{p-1} e\left(t\, a_{0 \ldots 0\, l\, 0 \ldots 0}\, x^d\right) = 0.$$

Thus the right-hand side of (9) vanishes in this case. This completes the proof of (8).

Lemma 2. If $t \not\equiv 0 \pmod p$ and $u \not\equiv 0 \pmod p$

$$(10)\quad \sum_{\deg f=d} e(tf(\underline{x})+uf(\underline{y})) = \begin{cases} p^k-1, & \text{if } tx_1^{i_1}\ldots x_n^{i_n}+uy_1^{i_1}\ldots y_n^{i_n} \equiv 0 \\ & \text{for all } i_1,\ldots,i_n \text{ satisfying} \\ & 0 \leqslant i_j \leqslant d \ (j=1,2,\ldots,n) \text{ and} \\ & i_1+\ldots+i_n=d \\ -1, & \text{otherwise.} \end{cases}$$

Proof. We have as before

$$\sum_{\deg f=d} e(tf(\underline{x})+uf(\underline{y}))+1$$

$$(11)\quad = \prod_{\substack{i_1,\ldots,i_n=0 \\ i_1+\ldots+i_n=d}}^{d} \left\{ \sum_{a_{i_1\ldots i_n}=0}^{p-1} e(a_{i_1\ldots i_n}(tx_1^{i_1}\ldots x_n^{i_n}+uy_1^{i_1}\ldots y_n^{i_n})) \right\}.$$

If $x_1, \ldots, x_n, y_1, \ldots, y_n$ are such that

$$tx_1^{i_1}\ldots x_n^{i_n}+uy_1^{i_1}\ldots y_n^{i_n} \equiv 0$$

are *all* $i_1, \ldots, i_n$ satisfying $0 \leqslant i_j \leqslant d$ $(j=1, 2, \ldots, n)$ and $i_1+\ldots+i_n=d$, then the right-hand side of (11) is just

$$\prod_{\substack{i_1,\ldots,i_n=0 \\ i_1+\ldots+i_n=d}}^{d} \left\{ \sum_{a_{i_1\ldots i_n}=0}^{p-1} 1 \right\} = p^k.$$

Otherwise, there is a n-tple $(i_1, \ldots, i_n)$ such that

$$tx_1^{i_1}\ldots x_n^{i_n}+uy_1^{i_1}\ldots y_n^{i_n} \not\equiv 0$$

and so for this

$$\sum_{a_{i_1\ldots i_n}=0}^{p-1} e(a_{i_1\ldots i_n}(tx_1^{i_1}\ldots x_n^{i_n}+uy_1^{i_1}\ldots y_n^{i_n}))=0,$$

which implies the vanishing of the right-hand side of (11). This completes the proof of (10).

Lemma 3. The number N of solutions $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ of the system of congruences

(12) $$t x_1^{i_1} \dots x_n^{i_n} + u y_1^{i_1} \dots y_n^{i_n} \equiv 0,$$

where the i_j $(j=1, 2, \dots, n)$ run through all integers satisfying

$0 \leqslant i_j \leqslant d$ and $i_1 + \dots + i_n = d$ and $t, u \not\equiv 0$, satisfies $N \leqslant d^n p^n$.

Proof. The number of solutions of (12) is less than or equal to the number of the system

(13) $$t x_i^d + u y_i^d \equiv 0 \ (i=1, 2, \dots, n).$$

The number of pairs (x_i, y_i) satisfying (13) is just

$$1 + (p-1)\, w,$$

where

$$w = \begin{cases} 0 & \text{if } (-tu^{-1})^{\frac{p-1}{(d,\, p-1)}} \not\equiv 1 \\ (d, p-1) & \text{if } (-tu^{-1})^{\frac{p-1}{(d,\, p-1)}} \equiv 1 \end{cases}$$

Hence the required number N satisfies

$$N \leqslant \left\{ 1 + (p \quad 1)\ w \right\}^n \leqslant d^n p^n .$$

Lemma 4.

$$\sum_{\deg f = d} \left(\sum_{\underline{x} \in C} \left\{ \frac{f(\underline{x})}{p} \right\} \right) = \tfrac{1}{2} p^{n+k} + O(p^{n+k-1}).$$

Proof. It was shown in [1] that

$$\sum_{\underline{x} \in C} \left\{ \frac{f(\underline{x})}{p} \right\} = \frac{1}{p^2} \sum_{r=1}^{p-1} r \sum_{\underline{x} \in C} \sum_{t=0}^{p-1} e(t(f(\underline{x}) - r))$$

so

$$\sum_{\deg f = d} \left(\sum_{\underline{x} \in C} \left\{ \frac{f(\underline{x})}{p} \right\} \right)$$

$$= \frac{1}{p^2} \sum_{r=1}^{p-1} r \sum_{t=0}^{p-1} e(-rt) \sum_{\underline{x} \in C} \sum_{\deg f = d} e(t\, f(\underline{x}))$$

$$= \frac{1}{p^2} \sum_{r=1}^{p-1} r \sum_{\underset{\sim}{x} \in C} \sum_{deg\, f = d} 1$$

$$+ \frac{1}{p^2} \sum_{r=1}^{p-1} r \sum_{t=1}^{p-1} e(-rt) \sum_{\underset{\sim}{x} \in C} \sum_{deg\, f = d} e(t.f(\underset{\sim}{x}))$$

$$= \frac{1}{p^2} \cdot \frac{(p-1)\,p}{2} \cdot p^n \cdot (p^k-1)$$

$$+ \frac{1}{p^2} \sum_{r=1}^{p-1} r \sum_{t=1}^{p-1} e(-rt) \left((p^k-1)-(p^n-1)\right),$$

by lemma 1. Now

$$\frac{1}{p^2} \sum_{r=1}^{p-1} r \sum_{t=1}^{p-1} e(-rt)\,(p^k-p^n) = -\frac{1}{p^2} \cdot \frac{(p-1)p}{2} \cdot (p^k-p^n)$$

so
$$\sum_{deg\, f = d} \left(\sum_{\underset{\sim}{x} \in C} \left\{ \frac{f(\underset{\sim}{x})}{p} \right\} \right) = \tfrac{1}{2}\,(p-1)\,p^{k-1}\,(p^n-1)$$

$$= \tfrac{1}{2}\,p^{n+k} + O(p^{n+k-1}).$$

Lemma 5.

$$\sum_{deg\, f = d} \left(\sum_{\underset{\sim}{x} \in C} \left\{ \frac{f(\underset{\sim}{x})}{p} \right\} \right)^2 = \frac{1}{4}\, p^{2n+k} + O(p^{2n+k-1}).$$

Proof

$$\left(\sum_{\underset{\sim}{x} \in C} \left\{ \frac{f(\underset{\sim}{x})}{p} \right\} \right)^2$$

$$= \left(\frac{1}{p^2} \sum_{r=1}^{p-1} r \sum_{\underset{\sim}{x} \in C} \sum_{t=0}^{p-1} e(t(f(\underset{\sim}{x})-r)) \right)^2$$

$$= \frac{1}{p^4} \sum_{r=1}^{p-1} \sum_{s=1}^{p-1} rs \sum_{\underset{\sim}{x} \in C} \sum_{\underset{\sim}{y} \in C} \sum_{t=0}^{p-1} \sum_{u=0}^{p-1}$$

$$e(t\,(f(\underset{\sim}{x})-r)+u(f(\underset{\sim}{y})-s))$$

so

$$\sum_{deg\, f=d} \left(\sum_{\underline{x} \in C} \left\{ \frac{f(\underline{x})}{p} \right\} \right)^2$$

$$= \frac{1}{p^4} \sum_{t=0}^{p-1} \sum_{u=0}^{p-1} \sum_{r=1}^{p-1} r\, e(-rt) \sum_{s=1}^{p-1} se\,(-su)$$

$$\sum_{\underline{x} \in C} \sum_{\underline{y} \in C} \sum_{deg\, f=d} e\,(t f(\underline{x}) + uf(\underline{y}))$$

$$= A_{00} + A_{01} + A_{10} + A_{11},$$

where firstly,

$$A_{00} = \frac{1}{p^4} \sum_{r=1}^{p-1} r \sum_{s=1}^{p-1} s \sum_{\underline{x} \in C} \sum_{\underline{y} \in C} \sum_{deg\, f=d} 1$$

$$= \frac{1}{p^4} \cdot \left(\frac{(p-1)p}{2}\right)^2 \cdot p^{2n} \cdot (p^k - 1)$$

$$= \frac{1}{4}(p-1)^2\, p^{2n-2}\, (p^k - 1)$$

$$= \frac{1}{4}\, p^{2n+k} + O(p^{2n+k-1}),$$

secondly,

$$A_{01} = A_{10} = \frac{1}{p^4} \sum_{u=1}^{p-1} \sum_{r=1}^{p-1} r \sum_{s=1}^{p-1} se(-su) \sum_{\underline{x} \in C} \sum_{\underline{y} \in C} \sum_{deg\, f=d} e(uf(\underline{y}))$$

$$= \frac{1}{p^4} \sum_{u=1}^{p-1} \frac{(p-1)p}{2} \sum_{s=1}^{p-1} se(-su) \cdot p^n \cdot \left\{ p^k - 1 - (p^n - 1) \right\}$$

(by lemma 1)

$$= \frac{1}{p^4} \frac{(p-1)p}{2} \cdot p^n \cdot (p^k - p^n) \sum_{s=1}^{p-1} s \sum_{u=1}^{p-1} e(-su)$$

$$= -\frac{1}{p^4} \left(\frac{(p-1)p}{2}\right)^2 p^n (p^k - p^n)$$

$$= -\frac{1}{4}(p-1)^2\, p^{2n-2}\, (p^{k-n} - 1) = O(p^{n+k-2}),$$

and finally,

$$A_{11}=\frac{1}{p^4}\sum_{t=1}^{p-1}\sum_{u=1}^{p-1}\sum_{r=1}^{p-1} re(-rt)\sum_{s=1}^{p-1} se(-su)$$

$$\sum_{\underline{x}\in C}\sum_{\underline{y}\in C}\sum_{deg\, f=d} e(tf(\underline{x})+uf(\underline{y})).$$

Let $A_{11}=A_{110}+A_{111}$,

where

$$A_{110}=\frac{1}{p^4}\sum_{t=1}^{p-1}\sum_{u=1}^{p-1}\sum_{r=1}^{p-1} re(-rt)\sum_{s=1}^{p-1} se(-su)$$

$$\sum_{\underline{x}\in C}\sum_{\underline{y}\in C}\sum_{\substack{deg\, f=d \\ tx_1^{i_1}\dots x_n^{i_n}+uy_1^{i_1}\dots y_n^{i_n}\equiv 0 \\ all\ i_1,\dots,i_n}} e(tf(x)+uf(y)),$$

so by lemmas 2 and 3 we have

$$|A_{110}|\leqslant\frac{1}{p^4}\sum_{t=1}^{p-1}\sum_{u=1}^{p-1}\sum_{r=1}^{p-1} r\sum_{s=1}^{p-1} s\,.\,d^n\,.\,p^n\,.\,(p^k-1)$$

$$=\frac{1}{p^4}(p-1)^2\left(\frac{(p-1)p}{2}\right)^2 d^n\ p^n\,(p^k-1)$$

$$\leqslant\frac{d^n}{4}\,.\,p^{n+k+2}\qquad i.e.\ A_{110}=O(p^{n+k+2}).$$

Now

$$A_{111}=\frac{1}{p^4}\sum_{t=1}^{p-1}\sum_{u=1}^{p-1}\sum_{r=1}^{p-1} re(-rt)\sum_{s=1}^{p-1} se(-st)$$

$$\sum_{\underline{x}\in C}\sum_{\underline{y}\in C}\sum_{\substack{deg\, f=d \\ tx_1^{i_1}\dots x_n^{i_n}+uy_1^{i_1}\dots y_n^{i_n}\not\equiv 0 \\ some\ i_1,\dots,i_n}} e(tf(\underline{x})+uf(\underline{y}))$$

so by lemma 2

$$|A_{111}| \leqslant \frac{1}{p^4} \sum_{t=1}^{p-1} \sum_{u=1}^{p-1} \sum_{r=1}^{p-1} r \sum_{s=1}^{p-1} s \cdot p^{2n}$$

$$= \frac{1}{p^4}(p-1)^2 \cdot \left(\frac{(p-1)p}{2}\right)^2 p^{2n}$$

$$\leqslant \frac{1}{4} p^{2n+2}, \qquad \text{i.e. } A_{111} = O(p^{2n+2}).$$

Hence

$$A_{11} = O(p^{n+k+2})$$

as $k > n$. This completes the proof of lemma 5.

Proof of Theorem. By lemmas 4 and 5

$$\sum_{\deg f=d} \left(\sum_{\underline{x} \in C} \left\{ \frac{f(\underline{x})}{p} \right\} - \frac{p^n}{2} \right)^2$$

$$= \sum_{\deg f=d} \left(\sum_{\underline{x} \in C} \left\{ \frac{f(\underline{x})}{p} \right\} \right)^2 - p^n \sum_{\deg f=d} \left(\sum_{\underline{x} \in C} \left\{ \frac{f(\underline{x})}{p} \right\} \right) + \frac{p^{2n}}{4} \sum_{\deg f=d} 1$$

$$= \frac{1}{4} p^{2n+k} + O(p^{2n+k-1}) - p^n \left(\frac{p^{n+k}}{2} + O(p^{n+k-1}) \right) + \frac{p^{2n+k}}{4} + O(p^{2n})$$

$$= O(p^{2n+k-1}).$$

Hence *almost all* homogeneous polynomials of degree $d \geqslant 1$ in $n \geqslant 3$ variables satisfy

$$\sum_{\underline{x} \in C} \left\{ \frac{f(\underline{x})}{p} \right\} = \tfrac{1}{2} p^n + O(p^{n-\frac{1}{2}}).$$

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