Chapter 7, Question 5

5. Let $K = \mathbb{Q}(\theta)$, where $\theta^3 - 9\theta - 6 = 0$. Prove that $\{1,\ \theta,\ \theta^2\}$ is an integral basis for K and that $d(K) = 2^3 \cdot 3^5$.

Solution. Let $K = \mathbb{Q}(\theta)$,where $\theta^3 - 9\theta - 6 = 0$. The monic polynomial $x^3 - 9x - 6 \in \mathbb{Z}[x]$ is 3-Eisenstein and so irreducible. Hence $[K : \mathbb{Q}] = 3$ and $\mathrm{irr}_{\mathbb{Q}}\theta = x^3 - 9x - 6$. Thus $\theta \in O_K$. Further $D(\theta) = -4(-9)^3 - 27(-6)^2 = 2^3 \cdot 3^5$. As $D(\theta)/d(K)$ is square we have

$$\frac{D(\theta)}{d(K)} = 1, 2^2, 3^2, 6^2, 9^2 \text{ or } 18^2.$$

Hence

$$d(K) = 2^3 \cdot 3^5, 2 \cdot 3^5, 2^3 \cdot 3^3, 2 \cdot 3^4, 2^3 \cdot 3 \text{ or } 2 \cdot 3.$$

As $d(K) \equiv 0$ or $1 \pmod 4$ we deduce that

$$d(K) = 2^3 \cdot 3^5, 2^3 \cdot 3^3 \text{ or } 2^3 \cdot 3.$$

We are going to show that

$$O_K = \mathbb{Z} + \mathbb{Z}\theta + \mathbb{Z}\theta^2,\ d(K) = 2^3 \cdot 3^5.$$

Suppose $d(K) = 2^3 \cdot 3^3$ or $2^3 \cdot 3$. Then

$$d(K) = \frac{D(\theta)}{3^2} \text{ or } \frac{D(\theta)}{3^4}$$

so there must exist an integer of K of the form $\dfrac{a + b\theta + c\theta^2}{3}$ $(a, b, c \in \mathbb{Z})$ with

$$\gcd(a, b, c, 3) = 1.$$

If $c \not\equiv 0 \pmod 3$ then $c = 3m \pm 1 (m \in \mathbb{Z})$ so

$$\frac{a + b\theta + c\theta^2}{3} = m\theta^2 \pm \left(\frac{\pm a \pm b\theta + c\theta^2}{3} \right)$$

and there is an integer of the form

$$\frac{A + B\theta + \theta^2}{3}.$$

If $c \equiv 0 \pmod 3$, say $c = 3m$ ($m \in \mathbb{Z}$),and $b \not\equiv 0 \pmod 3$ then $b = 3n \pm 1$ ($n \in \mathbb{Z}$) so

$$\frac{a + b\theta + c\theta^2}{3} = m\theta^2 + n\theta \pm \left(\frac{\pm a + \theta}{3}\right)$$

and there is an integer of the form

$$\frac{A + \theta}{3}.$$

If $c \equiv 0 \pmod 3$ and $b \equiv 0 \pmod 3$ then $(a, 3) = 1$ in which case

$$\frac{a + b\theta + c\theta^2}{3} = m\theta^2 + n\theta + \frac{a}{3}$$

so $\frac{a}{3} \in O_K$ and $\frac{a}{3} \in \mathbb{Q}$ giving $\frac{a}{3} \in \mathbb{Z}$, a contradiction, so this case cannot occur.

We show that there are no integers of K of the forms

$$\text{(I) } \frac{A + \theta}{3} \ (A \in \mathbb{Z}) \text{ and (II) } \frac{A + B\theta + \theta^2}{3} \ (A, B \in \mathbb{Z}).$$

(I) Let $\alpha = \dfrac{A + \theta}{3} \in O_K$. Then

$$\mathrm{irr}_{\mathbb{Q}}(\alpha) = x^3 - Ax^2 + \left(\frac{A^2}{3} - 1\right)x + \left(\frac{-A^3 + 9A - 6}{27}\right).$$

As $\alpha \in O_K$, $\mathrm{irr}_{\mathbb{Q}}\alpha \in \mathbb{Z}[x]$ so

$$\frac{A^2}{3} - 1 \in \mathbb{Z}, \quad \frac{-A^3 + 9A - 6}{27} \in \mathbb{Z}.$$

Hence $A = 3N$ for some $N \in \mathbb{Z}$ so

$$\frac{-27N^3 + 27N - 6}{27} \in \mathbb{Z},$$

a contradiction.

(II) Let $\alpha = \dfrac{A + B\theta + \theta^2}{3} \in O_K$. Then

$$\mathrm{irr}_{\mathbb{Q}}(\alpha) = x^3 - (A + 6)x^2 + \left(\frac{A^2}{3} + 4A - B^2 - 2B + 9\right)x$$

$$-\frac{1}{27}(A^3 + 6B^3 + 36 - 9AB^2 + 81A + 18A^2 - 54B - 18AB).$$

As $\alpha \in O_K,\ \mathrm{irr}_{\mathbb{Q}}\alpha \in \mathbb{Z}[x]$ so

$$\frac{A^2}{3} \in \mathbb{Z},\ \frac{A^3 + 6B^3 + 36 - 9AB^2 + 18A^2 - 18AB}{27} \in \mathbb{Z}.$$

Clearly $A = 3N$ for some $N \in \mathbb{Z}$ so

$$\frac{27N^3 + 6B^3 + 36 - 27NB^2 + 162N^2 - 54NB}{27} \in \mathbb{Z},$$

and thus

$$\frac{6B^3 + 36}{27} \in \mathbb{Z}. \tag{1}$$

Hence

$$2B^3 + 12 \equiv 0 \pmod 9$$

so $2B^3 \equiv 0 \pmod 3$ giving $B \equiv 0 \pmod 3$ contradicting with (1). Hence $d(K) \neq 2^3 \cdot 3^3$ or $2^3 \cdot 3$ so $d(K) = 2^3 \cdot 3^5$ and

$$D(1, \theta, \theta^2) = D(\theta) = 2^3 \cdot 3^5 = d(K)$$

so $\{1, \theta, \theta^2\}$ is an integral basis for K. ∎

February 23, 2004