Chapter 6, Question 20

20. Let $K = \mathbb{Q}(\theta)$, where $\theta^3 + 4\theta - 2 = 0$. Is $K = \mathbb{Q}(\theta + \theta^2)$?

Solution. The cubic polynomial $x^3 + 4x - 2$ is 2-Eisenstein and therefore irreducible. Hence $\mathrm{irr}_{\mathbb{Q}}(\theta) = x^3 + 4x - 2$ so that

$$[K : \mathbb{Q}] = [\mathbb{Q}(\theta) : \mathbb{Q}] = \deg(\mathrm{irr}_{\mathbb{Q}}(\theta)) = \deg(x^3 + 4x - 2) = 3.$$

Let $L = \mathbb{Q}(\theta+\theta^2)$. Clearly $\theta+\theta^2 \in K$ so that $L \subseteq K$. Thus $[L : \mathbb{Q}] \mid [K : \mathbb{Q}]$ so $[L : \mathbb{Q}] = 1$ or 3. If $[L : \mathbb{Q}] = 1$ then $L = \mathbb{Q}$ so $\theta + \theta^2 \in \mathbb{Q}$. Hence there exists a rational number a such that θ is a root of the quadratic polynomial x^2+x-a, contradicting that the minimal polynomial of θ over $\mathbb{Q}$ is of degree 3. Hence $[L : \mathbb{Q}] \neq 1$ and so $[L : \mathbb{Q}] = 3$. Thus $L = K$, that is $K = \mathbb{Q}(\theta+\theta^2)$. ■

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