CHAPTER 5, QUESTION 22

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22. Use Question 21 to prove that $[\mathbb{Q}(\theta) : \mathbb{Q}] = 4$, where θ is a root of $x^4 + 7x^2 + 5x + 1 = 0$.

Solution. In the notation of Question 21 we have $p = 5$, $a = 7$, $c = 4$. As

$$\left(\frac{a^2 - 4c}{p}\right) = \left(\frac{49 - 16}{5}\right) = \left(\frac{33}{5}\right) = \left(\frac{3}{5}\right) = -1,$$

$x^4 + 7x^2 + 5x + 4$ is irreducible by Question 21. Let θ be a root of $x^4 + 7x^2 + 5x + 4$ so that $\mathrm{irr}_{\mathbb{Q}}\theta = x^4 + 7x^2 + 5x + 4$. Hence

$$[\mathbb{Q}(\theta) : \mathbb{Q}] = \deg(\ \mathrm{irr}_{\mathbb{Q}}\theta) = 4. \qquad \blacksquare$$

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