CHAPTER 5, QUESTION 20

20. Let θ be a nonreal algebraic number. Prove that the complex conjugate $\bar{\theta}$ of θ is one of the conjugates of θ over $\mathbb{Q}$.
Solution. The minimalpolynomial of θ over $\mathbb{Q}$ is

$$\mathrm{irr}_{\mathbb{Q}}(\alpha) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{Q}[x].$$

Thus,

$$\theta^n + a_{n-1}\theta^{n-1} + \cdots + a_1\theta + a_0 = 0.$$

For $z \in \mathbb{C}$ we denote the complex conjugate of z by $\bar{z}$. As $a_0, \ldots, a_{n-1} \in \mathbb{Q} \subset \mathbb{R}$ we have

$$\bar{a_0} = a_0, \cdots, \overline{a_{n-1}} = a_{n-1}.$$

Also

$$\bar{\theta^k} = \bar{\theta}^k, \quad k = 1, 2, \ldots, n.$$

Hence

$$\begin{aligned} 0 = \bar{0} &= \overline{\theta^n + a_{n-1}\theta^{n-1} + \cdots + a_1\theta + a_0} \\ &= \overline{\theta^n} + \overline{a_{n-1}}\,\overline{\theta^{n-1}} + \cdots + \overline{a_1}\,\overline{\theta} + \overline{a_0} \\ &= \overline{\theta}^n + \overline{a_{n-1}}\,\overline{\theta}^{n-1} + \cdots + \overline{a_1}\,\overline{\theta} + \overline{a_0}. \end{aligned}$$

Thus $\bar{\theta}$ is a root of $\mathrm{irr}_{\mathbb{Q}}(\alpha)$ and so $\bar{\theta}$ is one of the conjugates of θ over $\mathbb{Q}$. ∎

June 23, 2004