CHAPTER 4, QUESTION 6

6. Let D be a principal ideal domain. Prove that D is integrally closed.

Solution. Let K =field of quotients of D. Let $\gamma \in K$ be integral over D. We must show that $\gamma \in D$. As $\gamma \in K$ we have $\gamma = \alpha/\beta$, where $\alpha \in D$ and $\beta(\neq 0) \in D$. As D is a principal ideal domain, $< \alpha, \beta > = < \delta >$ for some $\delta \in D$. Thus $\alpha = \alpha'\delta$ and $\beta = \beta'\delta$ for some $\alpha', \beta' \in D$. Also $\delta \in < \alpha, \beta >$ so $\delta = \theta\alpha + \phi\beta$ for some $\theta, \phi \in D$.Then $\delta = \theta\alpha'\delta + \phi\beta'\delta$. As $\beta \neq 0$ we have $\delta \neq 0$. Hence $1 = \theta\alpha' + \phi\beta'$. Thus $< \alpha', \beta' > = < 1 >$ and $\gamma = \alpha/\beta = \alpha'\delta/\beta'\delta = \alpha'\beta'$ with $\beta' = \beta/\delta \neq 0$. Relabel α' as α, β' as β, so

$$\gamma = \alpha/\beta, \quad < \alpha, \beta > = < 1 >, \quad \beta \neq 0, \quad \alpha, \beta \in D.$$

Since γ is integer over D, there exist $a_0, a_1, \ldots, a_{n-1} \in D$ such that

$$\gamma^n + a_{n-1}\gamma^{n-1} + \cdots + a_1\gamma + a_0 = 0.$$

Replacing γ by α/β, and multiplying both sides by β^n, we obtain

$$\alpha^n + a_{n-1}\alpha^{n-1}\beta + \cdots + a_1\alpha\beta^{n-1} + a_0\beta^n = 0.$$

Hence

$$\beta \mid \alpha^n.$$

Now $< \alpha, \beta > = < 1 >$ so we have

$$< \alpha, \beta >^n = < 1 >^n = < 1 >,$$

that is

$$< \alpha^n, \alpha^{n-1}\beta, \ldots, \alpha\beta^{n-1}, \beta^n > = < 1 > .$$

Hence

$$< \beta > < \alpha^n/\beta, \alpha^{n-1}, \ldots, \alpha\beta^{n-2}, \beta^{n-1} > = < 1 > .$$

Thus

$$< \alpha^n/\beta, \alpha^{n-1}, \ldots, \alpha\beta^{n-2}, \beta^{n-1} > = < \beta^{-1} > .$$

As the left hand side is an integral ideal of D, $< \beta^{-1} >$ must be an integral ideal of D. Thus $\beta^{-1} \in D$ and $\gamma = \alpha\beta^{-1} \in D$. This completes the proof that D is integrally closed. ∎

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