6. Prove that a subset N of an R-module M is a submodule of M if and only if
(i) $0 \in N$,
(ii) $n_1, n_2 \in N \Longrightarrow n_1 - n_2 \in N$,
(iii) $n \in N,\ r \in R \Longrightarrow rn \in N$.

Solution. Suppose that N is a submodule of M. Then N is a subgroup of M such that $rn \in N$ for all $r \in M$ and $n \in N$. This condition is just (iii), and (i), (ii) hold as N is a subgroup of M.

Now suppose that N is a subset of the R-module M satisfying (i), (ii), (iii). As M is an additive Abelian group and N is a subset of M satisfying (i) and (ii), N is a subgroup of M. Thus as N satisfies (iii), N is a submodule of M. ■