EXERCISES 3, QUESTION 3

3. Let $I_1 \subseteq I_2 \subseteq \ldots$ be an ascending chain of ideals in an integral domain D. Prove that $\bigcup_{n=1}^{\infty} I_n$ is an ideal in D.

Solution. Set

$$I = \bigcup_{n=1}^{\infty} I_n.$$

Let $a, b \in I$. Then $a \in I_r$ and $b \in I_s$ for some $r, s \in \mathbb{N}$. Let $m =$max(r, s). As $I_r \subseteq I_{r+1} \subseteq \ldots \subseteq I_m$ we have $a \in I_m$. As $I_s \subseteq I_{s+1} \subseteq \ldots \subseteq I_m$ we have $b \in I_m$. As I_m is an ideal, $a + b \in I_m$. Thus $a + b \in I$.

Now let $a \in I$ and $d \in D$. Then $a \in I_r$ for some $r \in \mathbb{N}$. As I_r is an ideal, $da \in I_r$. Hence $da \in I$.

This proves that $\bigcup_{n=1}^{\infty} I_n$ is an ideal of D. ■

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