EXERCISES 2, QUESTION 2

2. Prove Theorem 2.2.4.

Solution. Suppose first that $\mathbb{Z} + \mathbb{Z}(\frac{1+\sqrt{m}}{2})$ is Euclidean with respect to ϕ_m, where m is a squarefree integer $\equiv 1 \pmod 4$. Let $x, y \in \mathbb{Q}$. Then $x+y\sqrt{m} = (r + s\sqrt{m})/t$ for integers r, s, t with $t \neq 0$. As ϕ_m is a Euclidean function on $\mathbb{Z}+\mathbb{Z}(\frac{1+\sqrt{m}}{2})$ there exist $a+b(\frac{1+\sqrt{m}}{2}),\ c+d(\frac{1+\sqrt{m}}{2}) \in \mathbb{Z}+\mathbb{Z}(\frac{1+\sqrt{m}}{2})$ such that

$$r + s\sqrt{m} = t\left(a + b\left(\frac{1+\sqrt{m}}{2}\right)\right) + \left(c + d\left(\frac{1+\sqrt{m}}{2}\right)\right)$$

with

$$\phi_m\left(c + d\left(\frac{1+\sqrt{m}}{2}\right)\right) < \phi_m(t).$$

Hence

$$\begin{aligned}
\phi_m&\left((x+y\sqrt{m}) - \left(a + b\left(\frac{1+\sqrt{m}}{2}\right)\right)\right)\\
&= \phi_m\left(\frac{r+s\sqrt{m}}{t} - \left(a + b\left(\frac{1+\sqrt{m}}{2}\right)\right)\right)\\
&= \phi_m\left(\frac{r+s\sqrt{m} - t\left(a + b\left(\frac{1+\sqrt{m}}{2}\right)\right)}{t}\right)\\
&= \phi_m\left(\frac{c + d\left(\frac{1+\sqrt{m}}{2}\right)}{t}\right)\\
&= \frac{\phi_m\left(c + d(\frac{1+\sqrt{m}}{2})\right)}{\phi_m(t)} \quad \text{(by Lemma 2.2.1(d))}\\
&< 1,
\end{aligned}$$

as required.

Now suppose that for all $x, y \in \mathbb{Q}$ there exist $a, b \in \mathbb{Z}$ such that

$$\phi_m\left((x+y\sqrt{m}) - \left(a + b\left(\frac{1+\sqrt{m}}{2}\right)\right)\right) < 1. \tag{1}$$

To show that $\mathbb{Z}+\mathbb{Z}(\frac{1+\sqrt{m}}{2})$ is Euclidean with respect to ϕ_m, we must show that (2.1.1) and (2.1.2) hold. The inequality (2.1.1) holds in view of Lemma 2.2.1 (f). We now show that (2.1.2) holds. Let $r+s(\frac{1+\sqrt{m}}{2}),\ t+u(\frac{1+\sqrt{m}}{2}) \in \mathbb{Z}+\mathbb{Z}(\frac{1+\sqrt{m}}{2})$ with $t+u(\frac{1+\sqrt{m}}{2}) \neq 0$. Then

$$\frac{r+s(\frac{1+\sqrt{m}}{2})}{t+u(\frac{1+\sqrt{m}}{2})} = x + y\sqrt{m},$$

where

$$x = \frac{4rt+2ru+2st+(1-m)su}{4t^2+4tu+(1-m)u^2} \in \mathbb{Q}$$

and

$$y = \frac{2st-2ru}{4t^2+4tu+(1-m)u^2} \in \mathbb{Q}.$$

We note that

$$\begin{aligned} t+u(\frac{1+\sqrt{m}}{2}) \neq 0 &\Longrightarrow 2t+u+u\sqrt{m} \neq 0 \\ &\Longrightarrow (2t+u,u) \neq (0,0) \\ &\Longrightarrow (2t+u)^2 - mu^2 \neq 0 \text{ (as } m \text{ is squarefree)} \\ &\Longrightarrow 4t^2+4tu+(1-m)u^2 \neq 0. \end{aligned}$$

By (1) there exists $a+b(\frac{1+\sqrt{m}}{2}) \in \mathbb{Z}+\mathbb{Z}(\frac{1+\sqrt{m}}{2})$ such that

$$\phi_m\left((x+y\sqrt{m}) - \left(a+b\left(\frac{1+\sqrt{m}}{2}\right)\right)\right) < 1.$$

Set $c = r - at - bu(\frac{m-1}{4}) \in \mathbb{Z}$ and $d = s - au - bt - bu \in \mathbb{Z}$ and $d = s - au - bt - bu \in \mathbb{Z}$, so that

$$\begin{aligned} &c+d\left(\frac{1+\sqrt{m}}{2}\right) \\ &\quad = \left(r - at - bu\left(\frac{m-1}{4}\right)\right) + (s-au-bt-bw)\left(\frac{1+\sqrt{m}}{2}\right) \\ &\quad = \left(r+s\left(\frac{1+\sqrt{m}}{2}\right)\right) - \left(a+b\left(\frac{1+\sqrt{m}}{2}\right)\right)\left(t+u\left(\frac{1+\sqrt{m}}{2}\right)\right) \end{aligned}$$

as

$$\left(\frac{1+\sqrt{m}}{2}\right)^2 = \frac{m-1}{4} + \left(\frac{1+\sqrt{m}}{2}\right).$$

Hence

$$\begin{aligned} r &+ s\left(\frac{1+\sqrt{m}}{2}\right) \\ &= \left(a+b\left(\frac{1+\sqrt{m}}{2}\right)\right)\left(t+u\left(\frac{1+\sqrt{m}}{2}\right)\right) + \left(c+d\left(\frac{1+\sqrt{m}}{2}\right)\right) \end{aligned}$$

and

$$\begin{aligned} \phi_m &\left(c+d\left(\frac{1+\sqrt{m}}{2}\right)\right) \\ &= \phi_m\left(\left(r+s\left(\frac{1+\sqrt{m}}{2}\right)\right) - \left(a+b\left(\frac{1+\sqrt{m}}{2}\right)\right)\left(t+u\left(\frac{1+\sqrt{m}}{2}\right)\right)\right) \\ &= \phi_m\left((x+y\sqrt{m})\left(t+u\left(\frac{1+\sqrt{m}}{2}\right)\right)\right. \\ &\qquad\qquad \left. - \left(a+b\left(\frac{1+\sqrt{m}}{2}\right)\right)\left(t+u\left(\frac{1+\sqrt{m}}{2}\right)\right)\right) \\ &= \phi_m\left(\left(t+u\left(\frac{1+\sqrt{m}}{2}\right)\right)\left(x+y\sqrt{m} - \left(a+b\left(\frac{1+\sqrt{m}}{2}\right)\right)\right)\right) \\ &= \phi_m\left(t+u\left(\frac{1+\sqrt{m}}{2}\right)\right)\phi_m\left(x+y\sqrt{m} - \left(a+b\left(\frac{1+\sqrt{m}}{2}\right)\right)\right) \\ &< \phi_m\left(t+u\left(\frac{1+\sqrt{m}}{2}\right)\right), \end{aligned}$$

by Lemma 2.2.1(d), which completes the proof of (2.1.2). ∎

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