Chapter 10, Question 19

19. Let $K = \mathbb{Q}(\theta)$, where $\theta^3 - \theta - 1 = 0$. Prove that $\sqrt{\theta} \notin O_K$.

Solution. Suppose that

$$\sqrt{\theta} \in K = \mathbb{Q}(\theta).$$

Then there exist $a, b, c \in \mathbb{Q}$ such that

$$\sqrt{\theta} = a + b\theta + c\theta^2.$$

Hence

$$\begin{aligned} \theta &= (a + b\theta + c\theta^2)^2 \\ &= (a^2 + 2bc) + (2ab + c^2 + 2bc)\theta + (b^2 + c^2 + 2ac)\theta^2. \end{aligned}$$

Thus

$$a^2 + 2bc = 0, \tag{1}$$
$$2ab + 2bc + c^2 = 1, \tag{2}$$
$$b^2 + c^2 + 2ac = 0. \tag{3}$$

If $c = 0$ then (1) gives $a = 0$. This contradicts (2). Hence $c \neq 0$. Thus (1) gives

$$b = \frac{-a^2}{2c}.$$

Then (3) gives

$$\frac{a^4}{4c^2} + c^2 + 2ac = 0,$$

that is

$$\left(\frac{a}{c}\right)^4 + 8\left(\frac{a}{c}\right) + 4 = 0.$$

Hence $m = a/c$ is a rational root of the quadratic polynomial $x^4 + 8x + 4 \in \mathbb{Z}[x]$. Thus $m \in \mathbb{Z}$ and $m \mid 4$, so

$$m = \pm 1, \pm 2, \pm 4.$$

This is a contradiction as

$$m^4 + 8m + 4 = 13, -3, 36, 4, 292, 228,$$

according as $m = 1, -1, 2, -2, 4, -4$ respectively. ∎

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