CHAPTER 1, QUESTION 29

29. Give an example of an ideal in $\mathbb{Z}+\mathbb{Z}\sqrt{-6}$ which is not principal.

Solution. We show that the ideal $< 2, \sqrt{-6} >$ of $\mathbb{Z}+\mathbb{Z}\sqrt{-6}$ is not principal. Suppose on the contrary that $< 2, \sqrt{-6} >$ is principal. Then there exists $x+y\sqrt{-6} \in \mathbb{Z}+\mathbb{Z}\sqrt{-6}$ such that

$$< 2, \sqrt{-6} >=< x+y\sqrt{-6} > .$$

Now

$$2 \in < 2, \sqrt{-6} >=< x+y\sqrt{-6} >$$

so that there exists $a+b\sqrt{-6} \in \mathbb{Z}+\mathbb{Z}\sqrt{-6}$ such that

$$2 = (x+y\sqrt{-6})(a+b\sqrt{-6}).$$

Taking the modulus of both sides, we obtain

$$4 = (x^2+6y^2)(a^2+6b^2).$$

As $x^2+6y^2 \in \mathbb{N}$ we deduce that

$$x^2+6y^2 = 1,\ 2 \text{ or } 4.$$

If $x^2+6y^2 = 1$ then $x = \pm 1,\ y = 0$ so that

$$< 2, \sqrt{-6} >=< \pm 1 >=< 1 > .$$

Hence there exist $s+t\sqrt{-6} \in \mathbb{Z}+\mathbb{Z}\sqrt{-6}$ and $u+v\sqrt{-6} \in \mathbb{Z}+\mathbb{Z}\sqrt{-6}$ such that

$$1 = (s+t\sqrt{-6})2 + (u+v\sqrt{-6})\sqrt{-6}.$$

Equating real parts, we obtain

$$1 = 2s - 6v,$$

which is impossible. Clearly $x^2+6y^2 \neq 2$ for $x, y \in \mathbb{Z}$. Hence $x^2+6y^2 = 4$. Thus $x = \pm 2,\ y = 0$ so that

$$< 2, \sqrt{-6} >=< \pm 2 >=< 2 > .$$

Hence

$$\sqrt{-6} = 2(g + h\sqrt{-6})$$

for some $g+h\sqrt{-6} \in \mathbb{Z}+\mathbb{Z}\sqrt{-6}$. Equating imaginary parts, we obtain $2h = 1$, which is impossible. This completes the proof that the ideal $< 2, \sqrt{-6} >$ is not principal. $\blacksquare$

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