CHAPTER 1, QUESTION 13

13. Let A and B be ideals of an integral domain D. Prove that $AB \subseteq A \cap B$.

Solution. Let $\alpha \in AB$. Then there exist $a_1, \ldots, a_m \in A$ and $b_1, \ldots, b_m \in B$ such that

$$\alpha = a_1 b_1 + \cdots + a_m b_m.$$

For $i = 1, 2, \ldots, m$ we have

$$a_i b_i \in A, \text{ as } b_i \in D, \ a_i \in A \text{ and } A \text{ is an ideal,}$$

and

$$a_i b_i \in B, \text{ as } a_i \in D, \ b_i \in B \text{ and } B \text{ is an ideal.}$$

Thus

$$a_i b_i \in A \cap B, \ i = 1, 2, \ldots, m.$$

Since $A \cap B$ is an ideal by Question 10, we have $a_1 b_1 + \cdots + a_m b_m \in A \cap B$, so that $\alpha \in A \cap B$. We have shown that

$$AB \subseteq A \cap B.$$

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