CHAPTER 1, QUESTION 1

1. Prove that $U(\mathbb{Z}+\mathbb{Z}i) = \{\pm 1, \pm i\}$.

Solution. Let $\alpha \in U(\mathbb{Z}+\mathbb{Z}i)$. Then there exists $\beta \in \mathbb{Z}+\mathbb{Z}i$ such that

$$\alpha\beta = 1.$$

As $\alpha, \beta \in \mathbb{Z}+\mathbb{Z}i$ there exist $a, b, c, d \in \mathbb{Z}$ such that $\alpha = a + bi,\ \beta = c + di$. Thus

$$(a+bi)(c+di) = 1.$$

Equating real and imaginary parts, we obtain

$$ac - bd = 1,\ ad + bc = 0.$$

Thus

$$(a^2+b^2)(c^2+d^2) = (ac-bd)^2 + (ad+bc)^2 = 1.$$

As $a, b \in \mathbb{Z}$, a^2+b^2 is a positive integer dividing 1. Thus $a^2+b^2 = 1$ so that

$$(a,b) = (\pm 1, 0) \text{ or } (0, \pm 1).$$

Hence

$$\alpha = \pm 1 \text{ or } \pm i.$$

This proves that

$$U(\mathbb{Z}+\mathbb{Z}i) \subseteq \{1, -1, i, -i\}.$$

As

$$1 \cdot 1 = (-1)(-1) = i(-i) = -i(i) = 1,$$

we see that $1, -1, i, -i \in U(\mathbb{Z}+\mathbb{Z}i)$ so that

$$\{1, -1, i, -i\} \subseteq U(\mathbb{Z}+\mathbb{Z}i).$$

This completes the proof of

$$U(\mathbb{Z}+\mathbb{Z}i) = \{1, -1, i, -i\}. \qquad \blacksquare$$

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