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DEGENERATE STOCHASTIC CONTROL PROBLEMS WITH EXPONENTIAL COSTS AND WEAKLY COUPLED DYNAMICS: VISCOSITY SOLUTIONS AND A MAXIMUM PRINCIPLE*

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Abstract. This paper considers a class of optimization problems arising in wireless communication systems. We analyze the optimal control and the associated Hamilton–Jacobi–Bellman (HJB) equations. It turns out that the value function is a unique viscosity solution of the HJB equation in a certain function class. To deal with the fast growth condition of the value function in establishing uniqueness, we construct particular semiconvex/semiconcave approximations for the viscosity sub/supersolutions, and obtain a maximum principle on unbounded domains. The localized envelope function technique introduced in this paper permits an analysis of the uniqueness of viscosity solutions defined on unbounded domains in cases with very general growth conditions when combined with appropriate system dynamics. The optimization problem with state constraints is also considered.

Key words. degenerate stochastic control, power control, HJB equations, dynamic programming, viscosity solutions

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1. Introduction.

This paper is concerned with a class of optimization problems arising in power control for wireless communication systems, and forms a mathematical foundation for the results in [7, 8]. We will first formulate a class of degenerate stochastic control problems, which take the approach of regulating the state of a controlled process where an exogenous random parameter process is involved in the performance function, and then we use a communications application example to give a background illustration for the general formulation.

The random parameter process and the controlled process are denoted by $x_t \in \mathbb{R}^n$ and $p_t \in \mathbb{R}^n$, $t \in \mathbb{R}_+$, respectively. Suppose that x is modeled by the stochastic differential equation

$$dx = f(t,x)dt + \sigma(t,x)dw, \quad t \geq 0, \tag{1.1}$$

where f and σ are the drift and diffusion coefficients, respectively; w is an n dimensional standard Wiener process with covariance $Ew_t w_t^\tau = tI$; and the initial state x_0 is independent of $\{w_t,\ t \geq 0\}$ with finite exponential moment, i.e., $Ee^{2|x_0|} < \infty$.

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The process p is governed by the model

$$dp = g(t, x, p, u)dt, \quad t \geq 0, \tag{1.2}$$

where the component $g_i(t, x, p, u)$, $1 \leq i \leq n$, controls the size of the increment dp_i at the time instant t, $u \in \mathbb{R}^n$, $|u_i| \leq u_{i\,\max}$, $1 \leq i \leq n$. Without loss of generality we set $u_{i\,\max} = 1$, and we shall write

$$x = [x_1, \ldots, x_n]^\tau, \qquad p = [p_1, \ldots, p_n]^\tau, \qquad u = [u_1, \ldots, u_n]^\tau.$$

In the regulation of p, we introduce the following cost function:

$$J = E \int_0^T [p^\tau C(x)p + 2D^\tau(x)p]dt, \tag{1.3}$$

where $T < \infty$; $C(x)$ and $D(x)$ are an $n \times n$ positive definite matrix and an $n \times 1$ vector, respectively; and the components of $C(x)$ and $D(x)$ are exponential functions of linear combinations of x_i, $1 \leq i \leq n$. For simplicity, in this paper we take $C_{ij}(x) = c_{ij}e^{x_i + x_j}$, $D_i(x) = d_i e^{x_i} + s_i$ for $1 \leq i, j \leq n$. This particular structure of the weight coefficients indicates that in the cost function each p_i is directly associated with the parameter component x_i for $1 \leq i \leq n$. Specifically, an expansion of the cost integrand will produce entries in the form of $c_{ij}(e^{x_i}p_i)(e^{x_j}p_j)$, $d_i e^{x_i} p_i$, $s_i p_i$, $1 \leq i, j \leq n$. Intuitively, such a cost structure indicates that the relative weight of each p_i is influenced only by the process x_i. The more general case of expressing the components of $C(x)$ and $D(x)$ as exponential functions of general linear combinations of x_i, $1 \leq i \leq n$, can be considered without further difficulty. We will give the complete optimal control formulation in section 2, where the technical assumptions of weak coupling for the dynamics (1.1)–(1.2) will be introduced.

1.1. The stochastic power control example. We now briefly describe the motivating stochastic power control problem for lognormal fading channels. In an urban environment, due to long distance transmission and reflections, the power attenuations of wireless networks are described by lognormal random processes. Let $x_i(t)$, $1 \leq i \leq n$, denote the power attenuation (expressed in dBs and scaled to the natural logarithm basis) at the instant t of the ith mobile user, and let $\alpha_i(t) = e^{x_i(t)}$ denote the actual attenuation. Based upon the work in [1], the power attenuation dynamics are given as a special form of (1.1):

$$dx_i = -a_i(x_i + b_i)dt + \sigma_i dw_i, \quad t \geq 0,\ 1 \leq i \leq n. \tag{1.4}$$

In (1.4) the constants $a_i, b_i, \sigma_i > 0$, $1 \leq i \leq n$. See [1] for a physical interpretation of the parameters, and furthermore, an experimental justification of the lognormal attenuation modeling may be found in the communications literature [5] using discrete time measurements. In a network, at time t the ith mobile user sends its power $p_i(t)$, and the received power at the base station is $e^{x_i(t)}p_i(t)$. The mobile user has to adjust its power p_i in real time so that a certain communication quality of service (QoS) is maintained. In [6, 7] the adjustment of the (sent) power vector p for the n users is modeled by simply taking $g(t, x, p, u) = u$ in (1.2), which is called the rate adjustment model. Based upon the system signal-to-interference ratio (SIR) requirements, the following averaged integrated performance function,

$$J = E \int_0^T \left\{ \sum_{i=1}^n \left[e^{x_i}p_i - \mu_i \left(\sum_{j=1}^n e^{x_j}p_j + \eta \right) \right]^2 + \lambda \sum_{i=1}^n p_i \right\} dt, \tag{1.5}$$

was employed in [7, 8], where $\eta > 0$ is the system background noise intensity, $\lambda \geq 0$, and μ_i, $1 \leq i \leq n$, is a set of positive numbers determined from the SIR requirements. The resulting power control problem is to adjust u as a function of the system state (x, p) so that the above performance function is minimized.

1.2. The main contents and organization. The analysis in this paper treats a general class of performance functions that have an exponential growth rate with respect to x_i, $1 \leq i \leq n$; hence this analysis covers the cost function in (1.5) and differs from that appearing in most stochastic control problems in the literature, where linear or polynomial growth conditions usually pertain [3, 12]. Two novel features of the class of models (1.1)–(1.2) are (i) neither the drift nor the diffusion of the state subprocess x is subject to control, and hence x may be regarded as an exogenous signal, and (ii) further, the controlled state subprocess p has no diffusion part. Hence (1.1)–(1.2) gives rise to degenerate stochastic control systems. Optimization of such systems leads to degenerate Hamilton–Jacobi–Bellman (HJB) equations, which in general do not admit classical solutions [4, 12].

This paper deals with the mathematical control theoretic questions arising from the class of stochastic optimal control problems considered in [7, 8], where some approximation and numerical methods are proposed for implementation of the control laws. For the resulting degenerate HJB equations, we adopt viscosity solutions and show that the value function of the optimal control is a viscosity solution. To prove uniqueness of the viscosity solution, we develop a localized semiconvex/semiconcave approximation technique. Specifically, we introduce particular localized envelope functions on the unbounded domain to generate semiconvex/semiconcave approximations on any compact set. Compared to previous works [4, 12], by use of the set of envelope functions we can treat very rapid growth conditions, and we note that no Lipschitz or Hölder-type continuity assumption is required for the function class involved. It is worthwhile to note that the localized envelope functions may be applied to generate local semiconvex/semiconcave approximations for viscosity solutions in risk-sensitive stochastic control problems with degenerate diffusions in which the cost involves an exponential function and usually has a very rapid growth.

We also consider the optimal control subject to state constraints, which leads to the formulation of constrained viscosity solutions to the associated second order HJB equations; this part is parallel to [11], where a first order HJB equation is investigated. The paper is organized as follows: in section 2 we state the existence and uniqueness of the optimal control and show that the value function is a viscosity solution to a degenerate HJB equation; we then give two theorems as the main results about the solution of the HJB equation. Section 3 is devoted to introducing a class of semiconvex/semiconcave approximations for continuous functions; this technique enables us to treat viscosity solutions with rapid growth. In section 4, we analyze the HJB equation and prove a maximum principle by which it follows that the HJB equation has a unique viscosity solution in a certain function class. Section 5 considers the control problem subject to state constraints.

Finally, we remark that in the case when an additional control term is introduced to the state subprocess x to give mathematically more general dynamics, one can also derive an HJB equation for the corresponding optimal control problem, which is interesting in its own right, and the semiconvex/semiconcave approximations and uniqueness analysis procedure developed in sections 3 and 4 may still be carried out under appropriate conditions. However, without further conditions for the dynamics of x in the controlled case, in general the control problem needs to be formulated in a weak solution framework, and the resulting analysis is not in the scope of the present paper.

2. The optimal control and HJB equations. We define

$$z = \begin{pmatrix} x \\ p \end{pmatrix}, \qquad \psi = \begin{pmatrix} f \\ g \end{pmatrix}, \qquad G = \begin{pmatrix} \sigma \\ 0_{n\times n} \end{pmatrix}.$$

We now write (1.1) and (1.2) together in the vector form

$$dz = \psi dt + G dw, \quad t \geq 0. \tag{2.1}$$

In the following analysis we will denote the state variable by (x, p) or z, or in a mixing form; as we do in section 4, we may also write the arguments for the functions in (1.1)–(1.2) in a unified way in terms of (t, z). We write the integrand in (1.3) as $l(z) = l(x, p) = p^\tau C(x)p + 2D^\tau(x)p$. For notational clarity, hereafter we use x_t with a real-valued subscript t to denote the value of the vector process x at time t, and x_i with an integer subscript i to denote the ith component of x; the interpretation of the notation should be clear from the context. This convention also holds for other vector processes involved in the analysis.

The admissible control set is specified as

$$\mathcal{U} = \{u(\cdot) \mid u_t \text{ is adapted to } \sigma(z_s, s \leq t) \text{ and } u_t \in U \stackrel{\triangle}{=} [-1,1]^n \ \forall 0 \leq t \leq T\}.$$

As is stated in the introduction, the initial state vector is independent of the $n \times 1$ Wiener process w_t, $t \geq 0$; we make the additional assumption that p has a deterministic initial value p_0 at $t = 0$. Then it is easily verified that $\sigma(z_s,\ s \leq t) = \sigma(x_s,\ s \leq t)$. Define $\mathcal{L} = \{u(\cdot) \mid u \text{ is adapted to } \sigma(z_s,\ s \leq t), u_t \in \mathbb{R}^n \text{ and } E\int_0^T |u_s|^2 ds < \infty\}$. If we endow $\mathcal{L}$ with an inner product $\langle u, u'\rangle \stackrel{\triangle}{=} E\int_0^T u^\tau u' ds$ for $u, u' \in \mathcal{L}$, then $\mathcal{L}$ constitutes a Hilbert space with the induced norm $\|u\| = \langle u, u\rangle^{\frac{1}{2}} \geq 0$, $u \in \mathcal{L}$. Under this norm, $\mathcal{U}$ is a bounded, closed, and convex subset of $\mathcal{L}$. Finally, the cost associated with the system (2.1) and a control $u \in \mathcal{U}$ is specified to be

$$J(s, z, u) = E\left[\int_s^T l(z_t)dt | z_s = z\right], \quad z \in \mathbb{R}^{2n}, \tag{2.2}$$

where $s \in [0, T]$ is taken as the initial time of the system; further, we set the value function

$$v(s, z) = \inf_{u\in\mathcal{U}} J(s, z, u), \quad z \in \mathbb{R}^{2n},$$

and simply write $J(0, z, u)$ as $J(z, u)$. The following assumptions on the time interval $[0, T]$ will be used in our further analysis.

(H1) In (1.1)–(1.2), $f \in C([0,T]\times\mathbb{R}^n, \mathbb{R}^n)$, $\sigma \in C([0,T]\times\mathbb{R}^n, \mathbb{R}^{n\times n})$, $g \in C([0,T]\times \mathbb{R}^{3n}, \mathbb{R}^n)$ and f, σ, g satisfy a uniform Lipschitz condition; i.e., there exists a constant $C_0 > 0$ such that $|f(t,x)-f(s,y)| \leq C_0(|t-s|+|x-y|)$, $|\sigma(t,x)-\sigma(s,y)| \leq C_0(|t-s|+|x-y|)$, $|g(t,x,p,u)-g(s,x,q,u)| \leq C_0(|t-s|+|p-q|)$, and $|g(t,x,p,u)-g(t,0,p,u)| \leq C_0$ for all $t, s \in [0,T]$, $u \in U$, and $x, y, p, q \in \mathbb{R}^n$. In addition, there exists a constant C_σ such that $|\sigma_{ij}(t,x)| \leq C_\sigma$ for $(t,x) \in [0,T] \times \mathbb{R}^n$ and $1 \leq i, j \leq n$.

(H2) For $1 \leq i \leq n$, $f_i(x)$ can be written as $f_i(x) = -a_i(t)x_i + f_i^0(t,x)$, where $a_i(t) \geq 0$ for $t \in [0,T]$, and $\sup_{[0,T]\times\mathbb{R}^n} |f_i^0(t,x)| \leq C_{f^0}$ for a constant $C_{f^0} > 0$.

Throughout this paper we assume that (H1) holds. (H2) is used in Theorems 2.5 and 2.6 for proving uniqueness of the viscosity solution. Clearly (H2) holds for the lognormal fading channel model in the power control example.

Remark 1. Assumption (H1) ensures existence and uniqueness of the solution to (2.1) for any fixed $u \in \mathcal{U}$, where the Lipschitz condition with respect to t will be used to obtain certain estimates in the proof of uniqueness of the viscosity solution. Here σ is assumed to be bounded so as to lead to a finite cost for any initial state and admissible control u.

From (H1)–(H2) it is seen that the system model has the following important features: first, in the diffusion process x the evolution of x_i does not receive strong influence from the other state component x_k, $k \neq i$, in the sense that the cross term $f_i^0(t,x)$ is bounded by a constant; second, an arbitrary increase of x alone in the function $g(t,x,p,u)$ does not result in an unbounded increase in the magnitude of g, and hence x imposes only a relatively weak impact on the evolution of p. Due to the above features, we shall refer to the model (1.1)–(1.2) analyzed in this paper as having *weakly coupled dynamics,* and (H2) will be conveniently referred to as the weak coupling condition for x, which will be used to establish uniqueness of the viscosity solution.

PROPOSITION 2.1 (see [7, 8]). *Assuming in the control model* (1.2) *that* $g(t,x,p,u)$ *is linear in* p *and* u*, i.e., that there exist continuous matrix functions* A_t, B_t *on* $[0,T]$ *such that* $g(t,x,p,u) = A_t p + B_t u$*, then there exists an optimal control* $\widehat{u} \in \mathcal{U}$ *such that* $J(x_0,p_0,\widehat{u}) = \inf_{u\in\mathcal{U}} J(x_0,p_0,u)$*, where* (x_0,p_0) *is the initial state at time* $s=0$*; if, in addition,* B_t *is invertible for all* $t \in [0,T]$*, then the optimal control* $\widehat{u}$ *is unique and uniqueness holds in the following sense: if* $\widetilde{u} \in \mathcal{U}$ *is another control such that* $J(x_0,p_0,\widetilde{u}) = J(x_0,p_0,\widehat{u})$*, then* $P_\Omega(\widetilde{u}_s \neq \widehat{u}_s) > 0$ *only on a set of times* $s \in [0,T]$ *of Lebesgue measure zero, where* Ω *is the underlying probability sample space.*

PROPOSITION 2.2. *Assuming* (H1)–(H2)*, the value function* v *is continuous on* $[0,T] \times \mathbb{R}^{2n}$ *and*

$$|v(s,z)| \leq C\left[1 + \sum_{i=1}^{n} e^{4z_i} + \sum_{i=n+1}^{2n} z_i^4\right], \tag{2.3}$$

where $C > 0$ *is a constant independent of* (s,z).

Proof. The continuity of v can be established by use of (H1) and the continuous dependence of the cost (2.2) on the initial condition for the system (2.1) when $u \in \mathcal{U}$ is fixed. For an initial state $z_s = z$ and any fixed $u \in \mathcal{U}$, using (H2), we express $z_i(t)$, $1 \leq i \leq n$, in terms of $z_i|_{t=0}$ with a bounded term involving $z_k(s)$, $0 \leq s \leq t$, $k \neq i$, and get

$$\sup_{0\leq t\leq T} Ee^{4z_i(t)} \leq C_1\big(1 + e^{4z_i|_{t=0}}\big),$$

where $C_1 > 0$. By use of the structure of $C(x)$ and $D(x)$ in the cost integrand l, we obtain the estimates in a straightforward way,

$$\begin{aligned}|J(s,z,u)| &\leq E\int_s^T |l(z_t)|dt \leq E\int_s^T C_2\left[1 + \sum_{i=1}^{n} e^{4z_i(t)} + \sum_{i=n+1}^{2n} z_i^4(t)\right]dt\\ &\leq C_3\left[1 + \sum_{i=1}^{n} e^{4z_i} + \sum_{i=n+1}^{2n} z_i^4\right],\end{aligned}$$

for constants C_2, C_3 independent of (s,z), and (2.3) follows. $\square$

We see that in (2.1) the noise covariance matrix GG^τ is not of full rank. In general, under such a condition the corresponding stochastic optimal control problem does not admit classical solutions due to the degenerate nature of the arising HJB equations. Here we analyze viscosity solutions.

DEFINITION 2.3. *$\underline{v}(t,z) \in C([0,T]\times\mathbb{R}^{2n})$ is called a* viscosity subsolution *to the HJB equation*

$$0 = -\frac{\partial v}{\partial t} + \sup_{u\in U}\left\{-\frac{\partial^\tau v}{\partial z}\psi\right\} - \frac{1}{2}\mathrm{tr}\left(\frac{\partial^2 v}{\partial z^2}GG^\tau\right) - l, \qquad v|_{t=T} = h(z), \quad z\in\mathbb{R}^{2n}, \tag{2.4}$$

if $\underline{v}|_{t=T} \le h$, and for any $\varphi(t,z)\in C^{1,2}([0,T]\times\mathbb{R}^{2n})$, whenever $\underline{v}-\varphi$ takes a local maximum at $(t,z)\in[0,T)\times\mathbb{R}^{2n}$, we have

$$-\frac{\partial \varphi}{\partial t} + \sup_{u\in U}\left\{-\frac{\partial^\tau \varphi}{\partial z}\psi\right\} - \frac{1}{2}\mathrm{tr}\left(\frac{\partial^2 \varphi}{\partial z^2}GG^\tau\right) - l \le 0, \quad z\in\mathbb{R}^{2n}, \tag{2.5}$$

at (t,z). Here $\overline{v}(t,z)\in C([0,T]\times\mathbb{R}^{2n})$ is called a viscosity supersolution *to (2.4) if $\overline{v}|_{t=T}\ge h$, and in (2.5) we have an opposite inequality at (t,z), whenever $\overline{v}-\varphi$ takes a local minimum at $(t,z)\in[0,T)\times\mathbb{R}^{2n}$. Additionally, $v(t,z)$ is called a* viscosity solution *if it is both a viscosity subsolution and a viscosity supersolution.*

THEOREM 2.4. *The value function v is a viscosity solution to the HJB equation*

$$0 = -\frac{\partial v}{\partial t} + \sup_{u\in U}\left\{-\frac{\partial^\tau v}{\partial z}\psi\right\} - \frac{1}{2}\mathrm{tr}\left(\frac{\partial^2 v}{\partial z^2}GG^\tau\right) - l, \qquad v(T,z)=0. \tag{2.6}$$

Proof. The value function v is continuous (by Proposition 2.2) and satisfies the boundary condition in (2.6). Now, for any $\varphi(t,z)\in C^{1,2}([0,T]\times\mathbb{R}^{2n})$, suppose $v-\varphi$ has a local maximum at (s,z_0), $s<T$. We denote by $z^{(1)}$, $z^{(2)}$ the first n and last n, respectively, components of z. In the following proof, we assume that $\varphi(t,z)=0$ for all $z^{(1)}$ such that $|z^{(1)}-z_0^{(1)}|\ge C$ for a constant $C>0$; otherwise we can multiply $\varphi(t,z)$ by a C^∞ function $\zeta(z^{(1)})$ with compact support and $\zeta(z^{(1)})=1$ for $|z^{(1)}-z_0^{(1)}|\le\frac{C}{2}$. We take a constant control $u\in[-1,1]$ on $[s,T]$ to generate z_u with initial state $z_s=z_0$ and write $\Delta(t,z)=v(t,z)-\varphi(t,z)$. Since (s,z_0) is a local maximum point of $\Delta(t,z)$, we can find $\epsilon>0$ such that $\Delta(s_1,z)\le\Delta(s,z_0)$ for $|s_1-s|+|z-z_0|\le\epsilon$. For $s_1\in(s,T]$, $z_s=z_0$, write $1_{A^\epsilon}=1_{(|s_1-s|+|z_{s_1}-z_0|\ge\epsilon)}$. Then we get the lower bound estimate

$$\begin{aligned}
&E[\Delta(s,z_0)-\Delta(s_1,z_{s_1})]\\
&\quad= E[\Delta(s,z_0)-\Delta(s_1,z_{s_1})](1-1_{A^\epsilon}) + E[\Delta(s,z_0)-\Delta(s_1,z_{s_1})]1_{A^\epsilon}\\
&\quad\ge E[\Delta(s,z_0)-\Delta(s_1,z_{s_1})]1_{A^\epsilon} \stackrel{\triangle}{=} S_0,
\end{aligned}$$

and using basic estimates for the change of the value function with respect to different initial states (see, e.g., [12, 3] for standard techniques), it follows that

$$\begin{aligned}
|S_0| &= O\left(Ee^{2|z_{s_1}^{(1)}|}1_{A^\epsilon}\right)\\
&= O\left(Ee^{2|z_{s_1}^{(1)}|}1_{(|z_{s_1}^{(1)}-z_0^{(1)}|\ge\epsilon/2)}\right) &(2.7)\\
&= O(|s-s_1|^2) &(2.8)
\end{aligned}$$

when $s_1 \downarrow s$. Here we obtain (2.7) by the fact that $z^{(2)}_{s_1} \to z^{(2)}_0$ uniformly as $s_1 \downarrow s$, which follows from the Lipschitz and boundedness (w.r.t. increment in x) conditions for $g(t,x,p,u)$, and obtain the bound (2.8) using basic moment estimates for $|z^{(1)}_{s_1} - z^{(1)}_0|^2$. It follows from (2.8) that

$$\lim_{s_1 \downarrow s} \frac{1}{s_1 - s} E[\Delta(s,z_0) - \Delta(s_1, z_{s_1})] \geq 0. \tag{2.9}$$

However, for $s_1 \in (s,T]$ we also have

$$\begin{aligned} &\frac{1}{s_1 - s} E[\Delta(s,z_0) - \Delta(s_1, z_{s_1})] \\ &\quad \leq \frac{1}{s_1 - s} E\left[\int_s^{s_1} l(z_t)dt - \varphi(s,z_0) + \varphi(s_1, z_{s_1})\right] \\ &\quad \to \left[l + \frac{\partial \varphi}{\partial s} + \frac{\partial^\tau \varphi}{\partial z}\psi\bigg|_u + \frac{1}{2}\mathrm{tr}\left(\frac{\partial^2 \varphi}{\partial z^2} GG^\tau\right)\right]\bigg|_{(s,z_0)} \qquad \forall u \in U \end{aligned} \tag{2.10}$$

as $s_1 \downarrow s$, where we get the inequality by the principle of optimality, and obtain (2.10) by using Ito's formula to express $\varphi(s_1, z_{s_1})$ near (s, z_0) and then taking expectations. In the above, since v satisfies the growth condition in Proposition 2.2, $\varphi(t,z) = 0$ for $|z^{(1)} - z^{(1)}_0| \geq C$, all the expectations are finite. Therefore, for $z \in \mathbb{R}^{2n}$, by (2.9) and (2.10),

$$\frac{\partial \varphi}{\partial s} + \min_{u \in U}\left\{\frac{\partial^\tau \varphi}{\partial z}\psi\right\} + \frac{1}{2}\mathrm{tr}\left(\frac{\partial^2 \varphi}{\partial z^2} GG^\tau\right) + l \geq 0$$

at (s, z_0). On the other hand, if $v - \varphi$ has a local minimum at (s, z_0), $s < T$, then for any small $\varepsilon > 0$ we can choose sufficiently small $s_1 \in (s,T]$ and find a control $u \in \mathcal{U}$ generating z_u such that

$$\begin{aligned} &E\{v(s,z_0) - \varphi(s,z_0) - v(s_1, z_{s_1}) + \varphi(s_1, z_{s_1})\} \\ &\quad \geq E\left\{\int_s^{s_1} l(z_t)dt + \varphi(s_1, z_{s_1}) - \varphi(s, z_0)\right\} - \varepsilon(s_1 - s). \end{aligned} \tag{2.11}$$

Similar to (2.8), we also have

$$E[\Delta(s,z_0) - \Delta(s_1, z_{s_1})] \leq O(|s - s_1|^2),$$

which, together with (2.11) and Ito's formula, gives

$$\frac{\partial \varphi}{\partial s} + \min_{u \in U}\left\{\frac{\partial^\tau \varphi}{\partial z}\psi\right\} + \frac{1}{2}\mathrm{tr}\left(\frac{\partial^2 \varphi}{\partial z^2} GG^\tau\right) + l \leq 0$$

at (s, z_0), so that the value function v is a viscosity solution. □

To analyze uniqueness of the viscosity solution, we introduce the function class $\mathcal{G}$ such that each $W \in \mathcal{G}$ satisfies the following:

(i) $W \in C([0,T] \times \mathbb{R}^{2n})$, and

(ii) for any $W \in \mathcal{G}$, there exist $C, k_1, k_2 > 0$ such that $|W(t,z)| \leq C[\sum_{i=1}^n e^{k_1|z_i|} + \sum_{i=1}^{2n} |z_i|^{k_2}]$.

Notice that in condition (ii), the constants C, k_1, k_2 may take a different set of values for different $W \in \mathcal{G}$. By Proposition 2.2 and Theorem 2.4 it follows that the value function v is a viscosity solution to the HJB equation (2.6) in the class $\mathcal{G}$. We now state the uniqueness result for the viscosity solutions.

THEOREM 2.5. *Assuming that* (H1)–(H2) *hold, there exists a unique viscosity solution to* (2.6) *in the class* $\mathcal{G}$.

Here we state a general maximum principle on an unbounded domain for the HJB equation (2.6). By considering two possibly distinct viscosity solutions v_1 and v_2 and setting, respectively, $(v_1, v_2) = (\underline{v}, \overline{v})$ and $(v_2, v_1) = (\underline{v}, \overline{v})$ in Theorem 2.6, we obtain Theorem 2.5 as a corollary. The proof of the maximum principle is postponed to section 4.

THEOREM 2.6. *Assuming that* (H1)–(H2) *hold, if* $\underline{v}, \overline{v} \in \mathcal{G}$ *are the viscosity subsolution and supersolution to* (2.6), *respectively, and* $\sup_{\partial^* Q_0}(\underline{v} - \overline{v}) < \infty$, *then*

$$\sup_{Q_0}(\underline{v} - \overline{v}) = \sup_{\partial^* Q_0}(\underline{v} - \overline{v}), \tag{2.12}$$

where $Q_0 = [0, T] \times \mathbb{R}^{2n}$, $\partial^* Q_0 = \{(T, z) : z \in \mathbb{R}^{2n}\}$.

3. Semiconvex and semiconcave approximations on compact sets. To facilitate our analysis, write the Hamiltonian

$$\widetilde{H}(t, z, u, \xi, V) = -\xi^\tau \psi(t, z, u) - \frac{1}{2}\mathrm{tr}\{V G(t, z) G^\tau(t, z)\} - l(z), \tag{3.1}$$

$$H(t, z, \xi, V) = \sup_{u \in U} \widetilde{H}(t, z, u, \xi, V),$$

where $\xi \in \mathbb{R}^{2n}$, V is a $2n \times 2n$ real symmetric matrix, and the other terms are defined in section 2. Then the HJB equation (2.6) may be written as

$$0 = -v_t + H(t, z, v_z, v_{zz}), \tag{3.2}$$

$$v(T, z) = 0. \tag{3.3}$$

DEFINITION 3.1 (see [12]). *A real value function* $\varphi(x)$ *defined on a convex set* $Q \subset \mathbb{R}^m$ *is said to be semiconvex on* Q *if there exists a constant* $C > 0$ *such that* $\varphi(x) + C|x|^2$ *is convex;* $\varphi(x)$ *is semiconcave on* Q *if* $-\varphi(x)$ *is semiconvex on* Q.

DEFINITION 3.2. *A real value function* $\varphi(x)$ *defined on a convex set* $Q \subset \mathbb{R}^m$ *is said to be locally semiconvex on* Q *if for any* $y \in Q$ *there exists a convex neighborhood* N_y *(relative to* Q*) of* y *such that* $\varphi(x)$ *is semiconvex on* N_y.

PROPOSITION 3.3. *If* $\varphi(x)$ *is locally semiconvex on a convex compact set* Q, *then* $\varphi(x)$ *is semiconvex on* Q.

Proof. For any $y \in Q$, there exists a convex set N_y open relative to Q such that $y \in N_y$ and $\varphi(x)$ is semiconvex on N_y. Thus there exists $C_y > 0$ such that $\varphi(x) + C_y|x|^2$ is convex on N_y. Since $\{N_y,\ y \in Q\}$ is an open cover of Q, there exists a finite subcover denoted by $\{N_{y_i},\ 1 \le i \le k\}$. Take $C = \max_{1 \le i \le k} C_{y_i}$, and then obviously $\varphi(x) + C|x|^2 \stackrel{\triangle}{=} \widehat{\varphi}(x)$ is convex on each N_{y_i}, $1 \le i \le k$. Now for any $x_1, x_2 \in Q$, $0 \le \lambda \le 1$, we prove that $\widehat{\varphi}(\lambda x_1 + (1-\lambda)x_2) \le \lambda\widehat{\varphi}(x_1) + (1-\lambda)\widehat{\varphi}(x_2)$. It suffices to consider the case $0 < \lambda < 1$. First, from the collection $\{N_{y_i},\ 1 \le i \le k\}$ we select open sets, without loss of generality, denoted as $\mathcal{N} \stackrel{\triangle}{=} \{N_{y_i},\ i = 1, \ldots, m \le k\}$

such that $L \triangleq \{x : x = \alpha x_1 + (1-\alpha)x_2,\ 0 \le \alpha \le 1\} \subset \cup_{N_{y_i} \in \mathcal{N}} N_{y_i}$. For simplicity we consider the case $m = 2$ and $x_1 \in N_{y_1}$, $x_2 \in N_{y_2}$. The general case may be treated inductively. To avoid triviality, we assume that neither N_{y_1} nor N_{y_2} covers L individually, and then we can find $x_a \in L$, $x_a \neq x_\lambda \triangleq \lambda x_1 + (1-\lambda)x_2$ such that $x_a \in N_{y_1} \cap N_{y_2}$ and $x_a = c_1 x_1 + (1-c_1)x_2$, $0 < c_1 < 1$. Without loss of generality we assume that x_λ is between x_1 and x_a. Then we further choose $x_b \in N_{y_1} \cap N_{y_2}$ such that $x_b = c_2 x_1 + (1-c_2)x_2$ and x_b is between x_a and x_2. Now it is obvious that $0 < c_2 < c_1 < \lambda < 1$. It is straightforward to verify that

$$x_\lambda = \frac{\lambda - c_1}{1-c_1}x_1 + \frac{1-\lambda}{1-c_1}x_a, \qquad x_a = \frac{c_1 - c_2}{\lambda - c_2}x_\lambda + \frac{\lambda - c_1}{\lambda - c_2}x_b, \qquad x_b = \frac{c_2}{c_1}x_a + \frac{c_1 - c_2}{c_1}x_2.$$

Hence we have

$$\begin{aligned}
\widehat{\varphi}(x_\lambda) &\le \frac{\lambda - c_1}{1-c_1}\widehat{\varphi}(x_1) + \frac{1-\lambda}{1-c_1}\widehat{\varphi}(x_a),\\
\widehat{\varphi}(x_a) &\le \frac{c_1 - c_2}{\lambda - c_2}\widehat{\varphi}(x_\lambda) + \frac{\lambda - c_1}{\lambda - c_2}\widehat{\varphi}(x_b),\\
\widehat{\varphi}(x_b) &\le \frac{c_2}{c_1}\widehat{\varphi}(x_a) + \frac{c_1 - c_2}{c_1}\widehat{\varphi}(x_2),
\end{aligned}$$

where we get the first two inequalities and the last one by the convexity of $\widehat{\varphi}(x)$ on N_{y_1} and N_{y_2}, respectively. By a simple transformation with the above inequalities to eliminate $\widehat{\varphi}(x_a)$ and $\widehat{\varphi}(x_b)$, we get

$$\widehat{\varphi}(x_\lambda) \le \lambda \widehat{\varphi}(x_1) + (1-\lambda)\widehat{\varphi}(x_2).$$

By arbitrariness of x_1, x_2 in Q it follows that $\widehat{\varphi}(x)$ is convex on Q. This completes the proof. ◻

We adopt the semiconvex/semiconcave approximation technique of [12, 2, 9, 10], but due to the highly nonlinear growth condition of the class $\mathcal{G}$, we apply a particular localized technique to construct envelope functions to generate semiconvex/semiconcave approximations on any bounded domain. For any $W \in \mathcal{G}$, define the upper/lower envelope functions with $\eta \in (0, 1]$,

$$W^\eta(t,z) = \sup_{(s,w)\in B^\eta(t,z)} \left\{ W(s,w) - \frac{1}{2\eta^2}(|t-s|^2 + |z-w|^2) \right\}, \tag{3.4}$$

$$W_\eta(t,z) = \inf_{(s,w)\in B^\eta(t,z)} \left\{ W(s,w) + \frac{1}{2\eta^2}(|t-s|^2 + |z-w|^2) \right\}, \tag{3.5}$$

where $B^\eta(t,z)$ denotes the closed ball (relative to $[0,T] \times \mathbb{R}^{2n}$) centering (t,z) with radius η. As will be shown in the following lemma, our construction above will generate semiconvex/semiconcave approximations to a given continuous function on a compact set for small η.

LEMMA 3.4. *For any fixed $W \in \mathcal{G}$ and compact convex set $Q \subset [0,T] \times \mathbb{R}^{2n}$, there exists a positive constant $\eta_Q \le 1$ depending only on Q such that for all $\eta \in (0, \eta_Q]$, $W^\eta(t,z)$ is semiconvex on Q, and $W_\eta(t,z)$ is semiconcave on Q.*

Proof. Since any fixed $W \in \mathcal{G}$ is uniformly continuous and bounded on any compact set Q, there exists $\overline{\eta}_Q > 0$ depending only on Q, so that for all positive $\eta \le \overline{\eta}_Q$ and $(t, z) \in Q$,

$$W^\eta(t,z) = \sup_{(s,w)\in B^{\eta/2}(t,z)} \left\{ W(s,w) - \frac{1}{2\eta^2}[|t-s|^2 + |z-w|^2] \right\}. \tag{3.6}$$

Indeed, we can find $\overline{\eta}_Q > 0$ such that for all $\eta \le \overline{\eta}_Q$, $|W(s,w) - W(t,z)| \le \frac{1}{16}$ for $(s,w) \in B^\eta(t,z)$, where $(t,z) \in Q$. Then for any (s,w) satisfying $\frac{\eta^2}{4} \le |s-t|^2 + |w-z|^2 \le \eta^2$, we have

$$W(s,w) - \frac{1}{2\eta^2}(|s-t|^2 + |w-z|^2) \le W(t,z) + \frac{1}{16} - \frac{1}{2\eta^2}\frac{\eta^2}{4} < W(t,z),$$

and (3.6) follows. In the following we assume $\eta \le \overline{\eta}_Q$. Next we show that for any $(t_0, z_0) \in Q$, $W^\eta(t,z)$ is semiconvex on $B^{\eta/4}(t_0,z_0) \cap Q$. It suffices to show that $W^\eta(t,z) + \frac{1}{2\eta^2}(t^2 + |z|^2)$ is convex on $B^{\eta/4}(t_0,z_0) \cap Q$. Denote

$$R(s,w,t,z) = W(s,w) - \frac{1}{2\eta^2}(|t-s|^2 + |z-w|^2) + \frac{1}{2\eta^2}(t^2 + |z|^2).$$

If $(t_1,z_1), (t_2,z_2) \in B^{\eta/4}(t_0,z_0) \cap Q$, we have $(t_2,z_2) \in B^{\eta/2}(t_1,z_1)$. For any $\lambda \in [0,1]$, we denote $(t_\lambda, z_\lambda) = (\lambda t_1 + (1-\lambda)t_2, \lambda z_1 + (1-\lambda)z_2) \in Q$. It is obvious that $B^{\eta/2}(t_\lambda, z_\lambda) \subset B^\eta(t_1,z_1) \cap B^\eta(t_2,z_2)$. Then it follows that

$$\begin{aligned}
&W^\eta(t_\lambda, z_\lambda) + \frac{1}{2\eta^2}[t_\lambda^2 + |z_\lambda|^2] \\
&\quad = \sup_{(s,w)\in B^\eta(t_\lambda,z_\lambda)} R(s,w,t_\lambda,z_\lambda) = \sup_{(s,w)\in B^{\eta/2}(t_\lambda,z_\lambda)} R(s,w,t_\lambda,z_\lambda) \\
&\quad = \sup_{(s,w)\in B^{\eta/2}(t_\lambda,z_\lambda)} [\lambda R(s,w,t_1,z_1) + (1-\lambda)R(s,w,t_2,z_2)] \\
&\quad \le \sup_{(s,w)\in B^{\eta/2}(t_\lambda,z_\lambda)} \lambda R(s,w,t_1,z_1) + \sup_{(s,w)\in B^{\eta/2}(t_\lambda,z_\lambda)} (1-\lambda)R(s,w,t_2,z_2) \\
&\quad \le \sup_{(s,w)\in B^\eta(t_1,z_1)} \lambda R(s,w,t_1,z_1) + \sup_{(s,w)\in B^\eta(t_2,z_2)} (1-\lambda)R(s,w,t_2,z_2) \\
&\quad = \lambda \left[W^\eta(t_1,z_1) + \frac{1}{2\eta^2}(t_1^2 + |z_1|^2) \right] + (1-\lambda) \left[W^\eta(t_2,z_2) + \frac{1}{2\eta^2}(t_2^2 + |z_2|^2) \right].
\end{aligned}$$

Thus $W^\eta(t,z)$ is semiconvex on $B^{\eta/4}(t_0,z_0) \cap Q$. Further, by Proposition 3.3, $W^\eta(t,z)$ is semiconvex on Q. Similarly we can prove that $W_\eta(t,z)$ is semiconcave on Q for $\eta \in (0, \widetilde{\eta}_Q]$, where $\widetilde{\eta}_Q \le 1$ depends only on Q. The lemma follows by taking $\eta_Q = \min\{\overline{\eta}_Q, \widetilde{\eta}_Q\}$. ∎

We use an example to illustrate the construction of the semiconvex approximation to a given function.

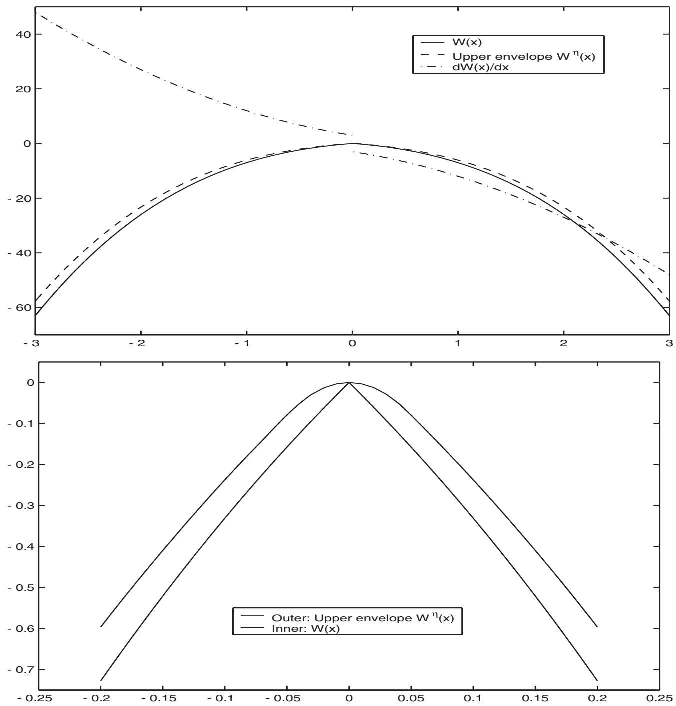

FIG. 3.1. *Semiconvex approximation with* $\eta = 0.125$. *Top: the curves in a large range. Bottom: the curves in the local region.*

Example 1. Consider a continuous function $W\colon \mathbb{R} \to \mathbb{R}$ defined as follows:

$$W(x) = \begin{cases} (x-1)^3 + 1 & \text{for } x \le 0, \\ -(x+1)^3 + 1 & \text{for } x > 0. \end{cases}$$

We take $0 < \eta \le 0.125$ and write

$$\theta(x) = 1 - x + \frac{1}{6\eta^2} - \sqrt{\left[1 - x + \frac{1}{6\eta^2}\right]^2 - (1-x)^2}, \quad x \le 0.$$

It is evident that the upper envelope function $W^\eta(x)$ is even on $\mathbb{R}$, and its value on $(-\infty, 0]$ is determined by

$$W^\eta(x) = \begin{cases} W(x+\eta) - \dfrac{1}{2} & \text{for } x \le 1 - \eta - \dfrac{1}{\sqrt{3\eta}}, \\ W(x + \theta(x)) - \dfrac{\theta^2(x)}{2\eta^2} & \text{for } 1 - \eta - \dfrac{1}{\sqrt{3\eta}} < x \le -3\eta^2, \\ W(0) - \dfrac{x^2}{2\eta^2} & \text{for } -3\eta^2 < x \le 0, \end{cases} \tag{3.7}$$

where $0 \le \theta(x) \le \eta \wedge |x|$ holds for $1 - \eta - \frac{1}{\sqrt{3\eta}} < x \le -3\eta^2$.

From Figure 3.1, it is seen that at $x = 0$ the first order derivative of $W(x)$ has a negative jump, which corresponds to a sharp turn at $x = 0$ on the function curve. After the semiconvexifying procedure, the sharp turn at $x = 0$ vanishes, as shown by the curve of $W^\eta(x)$.

We give a lemma which is parallel to the one in [12]. But here we do not make Lipschitz or Hölder-type continuity assumptions on W. For completeness we give the details.

LEMMA 3.5. *For $W \in \mathcal{G}$ and $\eta \in (0,1]$, W^η and W_η are equicontinuous (w.r.t. η) on any compact set $Q \subset [0,T] \times \mathbb{R}^{2n}$ and*

$$W^\eta(t,z) \le C\left[\sum_{i=1}^{n} e^{k_1|z_i|} + \sum_{i=1}^{2n} |z_i|^{k_2}\right], \tag{3.8}$$

$$W^\eta(t,z) = W(t_0,z_0) - \frac{1}{2\eta^2}(|t-t_0|^2 + |z-z_0|^2) \quad \text{for some } (t_0,z_0) \in B^\eta(t,z), \tag{3.9}$$

$$\frac{1}{2\eta^2}(|t-t_0|^2 + |z-z_0|^2) \to 0 \quad \text{uniformly on } Q \text{ as } \eta \to 0, \text{ and} \tag{3.10}$$

$$0 \le W^\eta(t,z) - W(t,z) \to 0 \quad \text{uniformly on } Q \text{ as } \eta \to 0, \tag{3.11}$$

where $C, k_1, k_2 > 0$ are constants independent of η. The estimates (3.8)–(3.10) *also hold when W^η is replaced by W_η, and*

$$0 \le W(t,z) - W_\eta(t,z) \to 0 \quad \text{uniformly on } Q \text{ as } \eta \to 0. \tag{3.12}$$

Proof. Inequality (3.8) follows from the definition of $\mathcal{G}$, and (3.9) is obvious. Moreover, by (3.9) we have

$$\frac{1}{2\eta^2}(|t-t_0|^2 + |z-z_0|^2) = W(t_0,z_0) - W^\eta(t,z) \le W(t_0,z_0) - W(t,z). \tag{3.13}$$

Since $|t-t_0| + |z-z_0| \to 0$ as $\eta \to 0$, by (3.13) and the uniform continuity of W on Q, (3.10) follows. The estimate (3.11) follows from (3.9) and (3.10). The equicontinuity of W^η (w.r.t. η) on Q can be established by (3.11) and the continuous dependence of W^η on $(\eta,t,z) \in [\varepsilon,1] \times Q$ for any $0 < \varepsilon \le 1$. The case of W_η can be treated similarly. $\square$

We define

$$H^\eta(t,z,\xi,V) = \inf_{(s,w)\in B^\eta(t,z)} \sup_{u\in U} \widetilde{H}(s,w,u,\xi,V), \tag{3.14}$$

$$H_\eta(t,z,\xi,V) = \sup_{(s,w)\in B^\eta(t,z)} \sup_{u\in U} \widetilde{H}(s,w,u,\xi,V). \tag{3.15}$$

Then it can be shown that H^η and H_η converge to $H(t,z,\xi,V)$ uniformly on any compact subset of $[0,T] \times \mathbb{R}^{2n} \times \mathbb{R}^{2n} \times S^{2n}$ as $\eta \to 0$, where S^{2n} denotes the set of $2n \times 2n$ real symmetric matrices. The following lemma can be proved by a method similar to that in [4, 9, 10]; the proof is omitted here. Notice that the viscosity sub/supersolution properties hold on a domain smaller than $[0,T] \times \mathbb{R}^{2n}$.

LEMMA 3.6. *If $\underline{v}$ ($\overline{v}$, respectively) is a viscosity subsolution (supersolution, respectively) to* (3.2) *on $[0,T] \times \mathbb{R}^{2n}$, then $\underline{v}^\eta$ ($\overline{v}_\eta$, respectively) is a viscosity subsolution (supersolution, respectively) to HJB equation A (B, respectively) on $[0,T-\eta] \times \mathbb{R}^{2n}$, where the HJB equations A and B are given by*

$$A: \begin{cases} -v_t + H^\eta(t,z,v_z,v_{zz}) = 0, \\ v(T-\eta,z) = \underline{v}^\eta(T-\eta,z), \end{cases} \qquad B: \begin{cases} -v_t + H_\eta(t,z,v_z,v_{zz}) = 0, \\ v(T-\eta,z) = \overline{v}_\eta(T-\eta,z). \end{cases}$$

In the above, $\underline{v}^\eta$ and $\overline{v}_\eta$ are defined by (3.4)–(3.5).

4. Proof of Theorem 2.6. In this section we give a proof of Theorem 2.6. We note that certain technical but standard arguments are not included here for reasons of economy of exposition; complete references to the detailed versions of these parts of the proof are supplied at appropriate places in the text.

We follow the method in [12, 4], employing the particular structure of the system dynamics, and will make necessary modifications. For the viscosity subsolution and supersolution $\underline{v}, \overline{v} \in \mathcal{G}$ we prove that

$$\sup_{Q_1}(\underline{v} - \overline{v}) = \sup_{\partial^* Q_0}(\underline{v} - \overline{v}) \stackrel{\triangle}{=} c_0 \quad \text{for } Q_1 = [T_1, T] \times \mathbb{R}^{2n}, \tag{4.1}$$

where $T_1 = T - \frac{1}{4\Delta}$, $\Delta = 25n(C_g + C_\sigma) + 10C_{f^0}$, C_g is a finite constant such that for g given in (1.2), $|g_i(t, x, p, u)| \leq C_g(1 + \sum_{k=1}^n |p_k|)$ for $t \in [0, T]$, $x, p \in \mathbb{R}^n$, $u \in U$, $1 \leq i \leq n$, and C_σ, C_{f^0} are given in assumptions (H1)–(H2) introduced in section 2. The maximum principle (2.12) follows by repeating the above procedure backward with time. Our proof by contradiction starts with the observation that if (4.1) is not true, there exists $(\hat{t}, \hat{z}) \in (T_1, T) \times \mathbb{R}^{2n}$ such that

$$\underline{v}(\hat{t}, \hat{z}) - \overline{v}(\hat{t}, \hat{z}) = c_0^+ > c_0. \tag{4.2}$$

We break the proof into several steps: (1) we construct a comparison function Λ depending on positive parameters α, β, ε, λ, and, based upon (4.2), Λ is used to induce a certain interior maximum; (2) using the viscosity sub/supersolution conditions, we get a set of inequalities at the interior maximum; and (3) we establish an inequality relation between α and β by taking appropriate vanishing subsequences of ε, λ, η, and this inequality relation is shown to lead to a contradiction. The weak coupling condition (H2) for x is used to obtain estimates used in Step 3 below.

Step 1 (constructing a comparison function and the interior maximum). To avoid introducing too many constants, we assume that $\underline{v}$ and $\overline{v}$ belong to the class $\mathcal{G}$ with associated constants $k_1 = k_2 = 4$. The more general case can be treated in exactly the same way. Now we define the comparison function

$$\begin{aligned}\Lambda(t, z, s, w) = {} & \frac{\alpha(2\mu T - t - s)}{2\mu T} \left\{ \sum_{i=1}^n \left[e^{5\sqrt{z_i^2+1}} + e^{5\sqrt{w_i^2+1}} \right] + \sum_{i=1}^{2n} (z_i^6 + w_i^6) \right\} \\ & - \beta(t + s) + \frac{1}{2\varepsilon}|t - s|^2 + \frac{1}{2\varepsilon}|z - w|^2 + \frac{\lambda}{t - T_1} + \frac{\lambda}{s - T_1},\end{aligned}$$

where α, β, ε, λ are all taken from $(0, 1]$; $\mu = 1 + \frac{1}{4T\Delta}$; $z, w \in \mathbb{R}^{2n}$; and $t, s \in (T_1, T]$. We write $\Phi(t, z, s, w) = \underline{v}^\eta(t, z) - \overline{v}_\eta(s, w) - \Lambda(t, z, s, w)$, where $\underline{v}^\eta$ and $\overline{v}_\eta$ are also in $\mathcal{G}$ by Lemma 3.5. Noticing that $\Phi \to -\infty$ as $t \wedge s \to T_1$ or $|z| + |w| \to \infty$, there exists (t_0, z_0, s_0, w_0) such that $\Phi(t_0, z_0, s_0, w_0) = \sup_{Q_1 \times Q_1} \Phi(t, z, s, w)$. By $\Phi(t_0, z_0, s_0, w_0) \geq \Phi(T, 0, T, 0)$, one can find a constant C_α depending only on α such that (see Remark 2)

$$|z_0| + |w_0| + \frac{1}{2\varepsilon}|t_0 - s_0|^2 + \frac{1}{2\varepsilon}|z_0 - w_0|^2 \leq C_\alpha \quad \text{and} \quad t_0, s_0 \in \left[T_1 + \frac{\lambda}{C_\alpha}, T \right]. \tag{4.3}$$

Combining $2\Phi(t_0, z_0, s_0, w_0) \geq \Phi(t_0, z_0, t_0, z_0) + \Phi(s_0, w_0, s_0, w_0)$, (4.3), and Lemma 3.5, we get for fixed $\alpha > 0$ (see Remark 3)

$$\frac{1}{2\varepsilon}|t_0 - s_0|^2 + \frac{1}{2\varepsilon}|z_0 - w_0|^2 \to 0 \quad \text{uniformly as } \varepsilon \to 0. \tag{4.4}$$

In this section, we take $\beta \in (0, \frac{c_0^+ - c_0}{4T})$. We further show that there exists $\alpha_0 > 0$ such that for $\alpha < \alpha_0$ and for sufficiently small r_0 (which may depend upon α) and $\eta \le r_0$, $\varepsilon \le r_0$, $\lambda \le r_0$, the maximum of Φ on Q_1 is attained at an interior point (t_0, z_0, s_0, w_0) of the set

$$Q_\alpha = \left\{ (t, z, s, w) : T_1 + \frac{\lambda}{2C_\alpha} \le t,\ s \le T - \eta,\ \text{and } |z|, |w| \le 2C_\alpha \right\}, \tag{4.5}$$

where C_α is determined in (4.3).

We begin by observing that $\Phi(t_0, z_0, s_0, w_0) \ge \Phi(\hat{t}, \hat{z}, \hat{t}, \hat{z})$ yields

$$\begin{aligned} \underline{v}^\eta(\hat{t}, \hat{z}) - \overline{v}_\eta(\hat{t}, \hat{z}) &\le \underline{v}^\eta(t_0, z_0) - \overline{v}_\eta(s_0, w_0) - \Lambda(t_0, z_0, s_0, w_0) + \Lambda(\hat{t}, \hat{z}, \hat{t}, \hat{z}) \\ &\le \underline{v}^\eta(t_0, z_0) - \overline{v}_\eta(s_0, w_0) + 2\beta T + \frac{2\lambda}{\hat{t} - T_1} \\ &\quad + 2\alpha \left[\sum_{i=1}^{n} e^{5\sqrt{\hat{z}_i^2 + 1}} + \sum_{i=1}^{2n} \hat{z}_i^6 \right]. \end{aligned} \tag{4.6}$$

Let $\mathbb{H}^\beta$ stand for the assertion that there exists α_0 such that when $\alpha \le \alpha_0$ and $\max\{\eta, \varepsilon, \lambda\} \le r_0$ for sufficiently small r_0, (t_0, z_0, s_0, w_0) is an interior point of Q_α in (4.5).

If $\mathbb{H}^\beta$ is not true, then there exists an arbitrarily small $\alpha \in (0, 1]$ such that for this fixed α we can select $\eta^{(k)}, \varepsilon^{(k)}, \lambda^{(k)} \to 0$ for which the resulting $(t_0^{(k)}, z_0^{(k)}, s_0^{(k)}, w_0^{(k)}) \notin \mathrm{Int}(Q_\alpha)$. By (4.3) it necessarily follows that $t_0^{(k)} \vee s_0^{(k)} \ge T - \eta^{(k)} \to T$ and (4.4) gives $|t_0^{(k)} - s_0^{(k)}| + |s_0^{(k)} - w_0^{(k)}| \to 0$. It is also clear that $(t_0^{(k)}, z_0^{(k)}, s_0^{(k)}, w_0^{(k)})$ is contained in a compact set determined by α. Then by selecting an appropriate subsequence of $(t_0^{(k)}, z_0^{(k)}, s_0^{(k)}, w_0^{(k)})$ and taking the limit in (4.6) along this subsequence, we get

$$\begin{aligned} \underline{v}(\hat{t}, \hat{z}) - \overline{v}(\hat{t}, \hat{z}) &\le \underline{v}(T, z^\alpha) - \overline{v}(T, z^\alpha) + \frac{c_0^+ - c_0}{2} + 2\alpha \left[\sum_{i=1}^{n} e^{5\sqrt{\hat{z}_i^2 + 1}} + \sum_{i=1}^{2n} \hat{z}_i^6 \right] \\ &\le \frac{c_0^+ + c_0}{2} + 2\alpha \left[\sum_{i=1}^{n} e^{5\sqrt{\hat{z}_i^2 + 1}} + \sum_{i=1}^{2n} \hat{z}_i^6 \right], \end{aligned} \tag{4.7}$$

where z^α denotes the common limit of the selected subsequences of $z_0^{(k)}$ and $w_0^{(k)}$. Sending $\alpha \to 0$, we get $\underline{v}(\hat{t}, \hat{z}) - \overline{v}(\hat{t}, \hat{z}) < c_0^+$, which contradicts (4.2); hence $\mathbb{H}^\beta$ holds. From the argument leading to (4.7) it is seen that α_0 can be chosen independently of β.

Step 2 (applying Ishii's lemma). Hereafter, we assume that $\beta < \frac{c_0^+ - c_0}{4T}$, $\alpha < \alpha_0$, and $\max\{\eta, \varepsilon, \lambda\} \le r_0$ are always satisfied and thus $\mathbb{H}^\beta$ holds. We assume Φ attains a strict maximum at (t_0, z_0, s_0, w_0); otherwise we replace Λ by $\Lambda + |t - t_0|^2 + |s - s_0|^2 + |z - z_0|^4 + |w - w_0|^4$. Following the derivations in [12, 9, 4] and using the interior maximum obtained in Step 1, the semiconvexity of $\underline{v}^\eta$, and the semiconcavity of $\overline{v}_\eta$ for $\eta \le \eta_{Q_\alpha}$ by Lemma 3.4, and by Lemma 3.6, we obtain the so-called Ishii's lemma; i.e., there exist $2n \times 2n$ symmetric matrices M_k, $k = 1, 2$, such that

$$-\Lambda_t(t_0, z_0, s_0, w_0) + H^\eta(t_0, z_0, \Lambda_z(t_0, z_0, s_0, w_0), M_1) \le 0, \tag{4.8}$$

$$\Lambda_s(t_0, z_0, s_0, w_0) + H_\eta(s_0, w_0, -\Lambda_w(t_0, z_0, s_0, w_0), M_2) \ge 0, \tag{4.9}$$

$$\begin{pmatrix} M_1 & 0 \\ 0 & -M_2 \end{pmatrix} \le \begin{pmatrix} \Lambda_{zz} & \Lambda_{zw} \\ \Lambda_{zw}^\tau & \Lambda_{ww} \end{pmatrix} \Big|_{(t_0, z_0, s_0, w_0)}. \tag{4.10}$$

We note that it is important to have $t_0 \vee s_0 < T-\eta$ in order to establish (4.8)–(4.9) by Lemma 3.6 and an approximation procedure (see, e.g., [4] for the case of a bounded domain). Now (4.8) and (4.9) yield

$$
\begin{aligned}
&-\Lambda_t(t_0,z_0,s_0,w_0)-\Lambda_s(t_0,z_0,s_0,w_0)\\
&\quad\le H_\eta(s_0,w_0,-\Lambda_w(t_0,z_0,s_0,w_0),M_2)-H^\eta(t_0,z_0,\Lambda_z(t_0,z_0,s_0,w_0),M_1).
\end{aligned}\tag{4.11}
$$

Step 3 (estimates for LHS and RHS of (4.11)). The final stage in our deduction of a contradiction from (4.2) involves estimates of the LHS and RHS of (4.11). The estimates for both sides of (4.11) are taken at (t_0,z_0,s_0,w_0), but for brevity we omit the subscript 0 for each variable. We have

$$
\begin{aligned}
\text{LHS of (4.11)} &= \frac{\alpha}{\mu T}\left[\sum_{i=1}^n\left(e^{5\sqrt{z_i^2+1}}+e^{5\sqrt{w_i^2+1}}\right)+\sum_{i=1}^n(z_i^6+w_i^6)\right]\\
&\quad+2\beta+\frac{\lambda}{(t-T_1)^2}+\frac{\lambda}{(s-T_1)^2}\\
&\ge\frac{\alpha}{\mu T}\left[\sum_{i=1}^n\left(e^{5\sqrt{z_i^2+1}}+e^{5\sqrt{w_i^2+1}}\right)+\sum_{i=1}^n(z_i^6+w_i^6)\right]+2\beta
\end{aligned}\tag{4.12}
$$

and

$$
\begin{aligned}
\text{RHS of (4.11)} &= \sup_{u\in U}[\Lambda_w^\tau(t,z,s,w)\psi(\widehat{s},\widehat{w},u)]-\sup_{u\in U}[-\Lambda_z^\tau(t,z,s,w)\psi(\widehat{t},\widehat{z},u)]\\
&\quad+\frac12\mathrm{tr}[G(\widehat{t},\widehat{z})G^\tau(\widehat{t},\widehat{z})M_1]-\frac12\mathrm{tr}[G(\widehat{s},\widehat{w})G^\tau(\widehat{s},\widehat{w})M_2]+l(\widehat{z})-l(\widehat{w})\\
&\le\sup_{u\in U}[\Lambda_w^\tau(t,z,s,w)\psi(\widehat{s},\widehat{w},u)+\Lambda_z^\tau(t,z,s,w)\psi(\widehat{t},\widehat{z},u)]\\
&\quad+\frac12\mathrm{tr}[G(\widehat{t},\widehat{z})G^\tau(\widehat{t},\widehat{z})M_1]-\frac12\mathrm{tr}[G(\widehat{s},\widehat{w})G^\tau(\widehat{s},\widehat{w})M_2]-l(\widehat{z})-l(\widehat{w}),
\end{aligned}
$$

which, together with (4.10) and (3.14)–(3.15), leads to

$$
\begin{aligned}
\text{RHS of (4.11)} &\le \sup_{u\in U}[\Lambda_w^\tau(t,z,s,w)\psi(\widehat{s},\widehat{w},u)+\Lambda_z^\tau(t,z,s,w)\psi(\widehat{t},\widehat{z},u)]\quad\left(\triangleq A_1\right)\\
&\quad+\frac{1}{2\varepsilon}\mathrm{tr}\{[G(\widehat{t},\widehat{z})-G(\widehat{s},\widehat{w})]^\tau[G(\widehat{t},\widehat{z})-G(\widehat{s},\widehat{w})]\}\quad\left(\triangleq A_2\right)\\
&\quad+\frac{\alpha(2\mu T-t-s)}{2\mu T}\\
&\quad\times\sum_{i,k=1}^n\frac12\left[\sigma_{ik}^2(\widehat{t},\widehat{z})(\Gamma''(z_i)+30z_i^4)\right.\\
&\qquad\left.+\sigma_{ik}^2(\widehat{s},\widehat{w})(\Gamma''(w_i)+30w_i^4)\right]\quad\left(\triangleq A_3\right)\\
&\quad+[l(\widehat{z})-l(\widehat{w})]\quad\left(\triangleq A_4\right)\\
&=A_1+A_2+A_3+A_4,
\end{aligned}\tag{4.13}
$$

where $\Gamma(r)\triangleq e^{5\sqrt{r^2+1}}$, $\Gamma''=\frac{d^2\Gamma}{dr^2}$ and $(\widehat{t}_0,\widehat{z}_0)\in B^\eta(t_0,z_0)$, $(\widehat{s}_0,\widehat{w}_0)\in B^\eta(s_0,w_0)$. Notice that the set $\mathcal{S}_{\eta,\varepsilon}=\{(t_0,z_0),(\widehat{t}_0,\widehat{z}_0),(s_0,w_0),(\widehat{s}_0,\widehat{w}_0)\}$ is contained in a compact set Q^*_α determined by α. For $0 < \varepsilon \le 1$ appearing in $\Lambda(t, z, s, w)$, there exists $\eta_\varepsilon > 0$ such that, for all $0 < \eta \le \eta_\varepsilon$,

$$\text{RHS of (4.11)} \le A^0_1 + A^0_2 + A^0_3 + A^0_4 + \varepsilon, \tag{4.14}$$

where, again without writing the subscript 0 for (t_0, z_0, s_0, w_0), we define

$$\begin{aligned}
A^0_1 &= \sup_{u\in U}[\Lambda^\tau_w(t, z, s, w)\psi(s, w, u) + \Lambda^\tau_z(t, z, s, w)\psi(t, z, u)],\\
A^0_2 &= \frac{1}{2\varepsilon}\mathrm{tr}\{[G(t, z) - G(s, w)]^\tau[G(t, z) - G(s, w)]\},\\
A^0_3 &= \frac{\alpha(2\mu T - t - s)}{2\mu T}\sum_{i,k=1}^{n}\frac{1}{2}[\sigma^2_{ik}(t, z)(\Gamma''(z_i) + 30z^4_i) + \sigma^2_{ik}(s, w)(\Gamma''(w_i) + 30w^4_i)],\\
A^0_4 &= l(z) - l(w).
\end{aligned}$$

Since $\mathcal{S}_{\eta,\varepsilon}$ is contained in Q^*_α and the diameter of $\mathcal{S}_{\eta,\varepsilon}$ tends to 0 as $\eta, \varepsilon \to 0$, by taking an appropriate sequence $(\eta^{(k)}, \varepsilon^{(k)}, \lambda^{(k)}) \to 0$ satisfying $\eta^{(k)} \le \eta_{\varepsilon^{(k)}}$, we get convergent sequences $(t^{(k)}_0, z^{(k)}_0)$, $(t^{(k)}_0, \hat{z}^{(k)}_0)$, $(s^{(k)}_0, w^{(k)}_0)$, $(s^{(k)}_0, \hat{w}^{(k)}_0) \to (\tilde{t}, \tilde{z})$ as $k \to \infty$. In the following we use the same C to denote different constants which are independent of α. Now we have the three relations

$$\limsup_{k\to\infty}\ \text{LHS of (4.11)}(\eta^{(k)}, \varepsilon^{(k)}, \lambda^{(k)}) \ge \frac{2\alpha}{\mu T}\left[\sum_{i=1}^{n} e^{5\sqrt{\tilde{z}^2_i+1}} + \sum_{i=1}^{2n}|\tilde{z}_i|^6\right] + 2\beta, \tag{4.15}$$

$$\lim_{k\to\infty}(A^0_2 + A^0_4)\big(\eta^{(k)}, \varepsilon^{(k)}, \lambda^{(k)}\big) = 0, \tag{4.16}$$

$$\limsup_{k\to\infty} A^0_3\big(\eta^{(k)}, \varepsilon^{(k)}, \lambda^{(k)}\big) \le \frac{n\alpha C_\sigma(\mu T - \tilde{t})}{\mu T}\sum_{i=1}^{n}\left(25e^{5\sqrt{\tilde{z}^2_i+1}} + 30|\tilde{z}_i|^4\right), \tag{4.17}$$

where (4.15) follows from (4.12), and (4.16) follows from the continuity of $l(z)$, the Lipschitz continuity of $G(t, z)$ by assumption (H1), and (4.4). We proceed to analyze A^0_1:

$$\begin{aligned}
A^0_1 \le \sup_{u\in U}&\sum_{i=n+1}^{2n}[\Lambda_{z_i}(t, z, s, w)\psi_i(t, z, u) + \Lambda_{w_i}(t, z, s, w)\psi_i(s, w, u)]\\
&+ \sum_{i=1}^{n}[\Lambda_{z_i}(t, z, s, w)f_i(t, z) + \Lambda_{w_i}(t, z, s, w)f_i(s, w)] \stackrel{\triangle}{=} A^0_{11} + A^0_{12}.
\end{aligned}$$

Then by (H1), (4.4), and recalling $|g_i(t, x, p, u)| \le C_g(1 + \sum_{k=1}^n |p_k|)$ for $t \in [0, T]$, $u \in U$, we have

$$\limsup_{k\to\infty} A^0_{11}\big(\eta^{(k)}, \varepsilon^{(k)}, \lambda^{(k)}\big) \le \frac{\alpha(\mu T - \tilde{t})}{\mu T}\sum_{i=n+1}^{2n} 12C_g\left[2n|\tilde{z}_i|^6 + |\tilde{z}_i|^5\right]. \tag{4.18}$$

Now we employ $a_i(t) \ge 0$ for $t \in [0, T]$ in the weak coupling condition (H2), and the Lipschitz property of $f_i(t, z) = a_i(t)z_i + f^0_i(t, z)$ by (H1) to obtain

$$
\begin{aligned}
A_{12}^0 = {} & \frac{\alpha(2\mu T - t - s)}{2\mu T} \\
& \times \sum_{i=1}^n \Bigg\{ \left[\frac{5z_i}{\sqrt{z_i^2+1}} e^{5\sqrt{z_i^2+1}} + 6z_i^5 + \frac{z_i - w_i}{\varepsilon} \right] [-a_i(t)z_i + f_i^0(t,z)] \\
& \qquad + \left[\frac{5w_i}{\sqrt{w_i^2+1}} e^{5\sqrt{w_i^2+1}} + 6w_i^5 + \frac{w_i - z_i}{\varepsilon} \right] [-a_i(s)w_i + f_i^0(s,w)] \Bigg\} \\
\le {} & \frac{\alpha(2\mu T - t - s)}{2\mu T} \sum_{i=1}^n \Bigg\{ \left[\frac{5z_i}{\sqrt{z_i^2+1}} e^{5\sqrt{z_i^2+1}} + 6z_i^5 \right] f_i^0(t,z) \\
& \qquad + \left[\frac{5w_i}{\sqrt{w_i^2+1}} e^{5\sqrt{w_i^2+1}} + 6w_i^5 \right] f_i^0(s,w) \Bigg\} \\
& + O\left(\frac{|t-s|^2}{\varepsilon} + \frac{|z-w|^2}{\varepsilon} \right).
\end{aligned} \tag{4.19}
$$

Hence, invoking (4.4), it follows that

$$
\limsup_{k\to\infty} A_{12}^0(\eta^{(k)}, \varepsilon^{(k)}, \lambda^{(k)}) \le \frac{\alpha C_{f^0}(\mu T - \tilde{t})}{\mu T} \sum_{i=1}^n \left[10 e^{5\sqrt{\tilde{z}_i^2+1}} + 12|\tilde{z}_i|^5 \right], \tag{4.20}
$$

which, together with (4.16)–(4.18), gives

$$
\begin{aligned}
& \limsup_{k\to\infty} \text{ RHS of (4.11) } (\eta^{(k)}, \varepsilon^{(k)}, \lambda^{(k)}) \\
& \quad \le \frac{[10C_{f^0} + 25n(C_\sigma + C_g)]\alpha(\mu T - \tilde{t})}{\mu T} \left[\sum_{i=1}^n e^{5\sqrt{\tilde{z}_i^2+1}} + \sum_{i=1}^{2n} |\tilde{z}_i|^6 + C \right] \\
& \quad \le \frac{\alpha}{2\mu T} \left[\sum_{i=1}^n e^{5\sqrt{\tilde{z}_i^2+1}} + \sum_{i=1}^{2n} |\tilde{z}_i|^6 + C \right],
\end{aligned} \tag{4.21}
$$

where C is independent of α. Hence it follows from (4.11), (4.15), and (4.21) that

$$
2\beta \le -\frac{3\alpha}{2\mu T} \left\{ \sum_{i=1}^n e^{5\sqrt{\tilde{z}_i^2+1}} + \sum_{i=1}^{2n} |\tilde{z}_i|^6 \right\} + \alpha C \le \alpha C. \tag{4.22}
$$

We recall from Step 1 that $\beta \le 1$ can take a strictly positive value from the interval $(0, \frac{c_0^+ - c_0}{4T})$ and $\alpha \in (0, \alpha_0)$. Letting $\alpha \to 0$ in (4.22) yields $\beta \le 0$, which is a contradiction to $\beta \in (0, \frac{c_0^+ - c_0}{4T})$, and this completes the proof.

Remark 2. By $\Phi(t_0, z_0, s_0, w_0) \ge \Phi(T, 0, T, 0)$ and $|\underline{v} - \overline{v}| = o([\sum_{i=1}^n (e^{5|z_i|} + e^{5|w_i|}) + \sum_{i=1}^{2n}(z_i^6 + w_i^6)])$, there exist $\delta_\alpha > 0$, $C > 0$ such that

$$
\begin{aligned}
& \frac{1}{2\varepsilon}|t_0 - s_0|^2 + \frac{1}{2\varepsilon}|z_0 - w_0|^2 + \frac{\lambda}{t_0 - T_1} + \frac{\lambda}{s_0 - T_1} \\
& \qquad + \delta_\alpha \left[\sum_{i=1}^n \left(e^{5\sqrt{1+z_{0,i}^2}} + e^{5\sqrt{1+w_{0,i}^2}} \right) + \sum_{i=1}^{2n} (z_{0,i}^6 + w_{0,i}^6) \right] \le C.
\end{aligned}
$$

Then (4.3) follows readily.

Remark 3. By expanding $2\Phi(t_0, z_0, s_0, w_0) \geq \Phi(t_0, z_0, t_0, z_0) + \Phi(s_0, w_0, s_0, w_0)$ using all the individual terms, it can be shown that $\frac{1}{2\varepsilon}|t_0 - s_0|^2 + \frac{1}{2\varepsilon}|z_0 - w_0|^2$ is dominated by a continuous function $F(t_0, z_0, s_0, w_0)$, which goes to zero as $|t_0 - s_0| + |z_0 - w_0| \to 0$, which in turn follows from (4.3) when $\varepsilon \to 0$.

Remark 4. The proof of the theorem is based upon the methods in [12, 9, 10, 2]. Since here we deal with the function class $\mathcal{G}$ with a highly nonlinear growth condition on an unbounded domain, a localized semiconvex/semiconcave approximation technique is devised. The particular structure of the system dynamics also plays an important role in the proof of uniqueness, and in general it is more difficult to obtain uniqueness results under more general dynamics and the above fast growth condition. It is seen that the weak coupling feature of the dynamics of the state subprocess x is crucial for the above proof, and furthermore, when there exists an $a_i < 0$ (see assumption (H2)), the estimate (4.19) would not be valid.

5. Control with state constraints. In this section we consider the case when the state subprocess p is subject to constraints; i.e., the trajectory of each p_i must be maintained to be in a certain range. We term this situation as optimization under hard constraints. In [11] the author considered a deterministic model and obtained a constrained viscosity solution formulation for a first order HJB equation. Now due to the exogenous subprocess x, we come up with a second order HJB equation, and we will develop a similar formulation. Suppose that $u \in U$, where U is a compact convex set in $\mathbb{R}^n$, and that p satisfies $p_i \in [0, \overline{P}_i]$, where $\overline{P}_i$ is the upper bound. For simplicity we take $U = [-1, 1]^n$ and $\overline{P}_i = \infty$. For any fixed initial value $p_0 \geq 0$ (i.e., each $(p_0)_i \geq 0$), define the admissible control set

$$\mathcal{U}^{p_0} = \{u(\cdot) \mid u \text{ is adapted to } \sigma(z_s,\ s \leq t),\ u(t) \in U,$$
$$\text{and } P_\Omega(p_i(t) \geq 0 \ \text{ for all } 0 \leq t \leq T) = 1,\ 1 \leq i \leq n\}.$$

In this section we consider the simple case of

$$g(t, x, p, u) = u.$$

Under the admissible control set $\mathcal{U}^{p_0}$, we will use the notation of section 2, for which the interpretation should be clear, and in the following we also use $\mathcal{U}^{p_0}$ with any initial time $s \leq T$. It is evident that $\mathcal{U}^{p_0}$ is a convex set. Under the norm $\|\cdot\|$ on $\mathcal{L}$ defined in section 2, $\mathcal{U}^{p_0}$ is also closed. Indeed, if $\|u^{(k)} - u\| \to 0$ as $k \to \infty$, where $u^{(k)} \in \mathcal{U}^{p_0}$, one can show that u will also generate positive p trajectories with probability 1 with initial value p_0. Thus $u \in \mathcal{U}^{p_0}$. As in the state unconstrained case, one can prove existence and uniqueness of the optimal control. Write

$$Q_T^0 = [0, T) \times \mathbb{R}^n \times (0, \infty)^n, \quad Q_T = [0, T) \times \mathbb{R}^n \times [0, \infty)^n,$$
$$\overline{Q}_T = [0, T] \times \mathbb{R}^n \times [0, \infty)^n.$$

We consider the HJB equation

$$0 = -\frac{\partial v}{\partial t} + \sup_{u \in U} \left\{ -\frac{\partial^\tau v}{\partial z} \psi \right\} - \frac{1}{2} \mathrm{tr} \left(\frac{\partial^2 v}{\partial z^2} G G^\tau \right) - l, \tag{5.1}$$
$$v|_{t=T} = 0,$$

where $(t, z) = (t, x, p) \in \overline{Q}_T$.

DEFINITION 5.1. *$v(t,z) \in C(\overline{Q}_T)$ is called a* constrained viscosity solution *to* (5.1) *if* (i) *$v|_{t=T} = 0$ and, for any $\varphi(t,z) \in C^{1,2}(\overline{Q}_T)$, whenever $v - \varphi$ takes a local maximum at $(t,z) \in Q_T^0$, we have*

$$-\frac{\partial \varphi}{\partial t} + \sup_{u \in U} \left\{ -\frac{\partial^\tau \varphi}{\partial z} \psi \right\} - \frac{1}{2} \mathrm{tr} \left(\frac{\partial^2 \varphi}{\partial z^2} G G^\tau \right) - l \leq 0, \quad z \in \mathbb{R}^{2n}, \tag{5.2}$$

at (t,z), and (ii) *for any $\varphi(t,z) \in C^{1,2}(\overline{Q}_T)$, whenever $v - \varphi$ takes a local minimum at $(t,z) \in Q_T$, in* (5.2) *we have an opposite inequality at (t,z). In short, we term the constrained viscosity solution $v(t,z) \in C(\overline{Q}_T)$ as a viscosity subsolution on Q_T^0 and a viscosity supersolution on Q_T.*

Remark 5. Conditions (i) and (ii) hold on Q_T^0 and Q_T, respectively. Here we give a heuristic interpretation on how the state constraints are captured by condition (ii). Suppose that $v - \varphi$ attains a minimum at $(\bar{t}, \overline{x}, \overline{p})$, where v is the value function and satisfies (5.1) at $(\bar{t}, \overline{x}, \overline{p})$ with classical derivatives, i.e.,

$$0 = -\frac{\partial v}{\partial t} + \left\{ -\frac{\partial^\tau v}{\partial z} \psi \right\} \bigg|_{u = \widehat{u}} - \frac{1}{2} \mathrm{tr} \left(\frac{\partial^2 v}{\partial z^2} G G^\tau \right) - l. \tag{5.3}$$

In addition, we assume that $\widehat{u}$ is admissible w.r.t. $(\overline{x}, \overline{p})$. Here $\bar{t} \in [0,T)$ and $\overline{p}$ lies on the boundary of $[0,\infty)^n$. By the necessary condition for a minimum, at $(\bar{t}, \overline{x}, \overline{p})$, we have

$$v_t - \varphi_t \geq 0, \quad v_{x_i} - \varphi_{x_i} = 0, \quad v_{x_i x_i} - \varphi_{x_i x_i} \geq 0, \quad 1 \leq i \leq n, \tag{5.4}$$

where the first inequality becomes equality when $\bar{t} \in (0,T)$. Since $\overline{p}$ is on the boundary of $[0,T)^n$, we can find an index set I such that $\overline{p}_i = 0$ when $i \in I$, and $\overline{p}_i > 0$ when $i \in \{1,\ldots,n\} \backslash I$. Again, by the minimum property at $(\bar{t}, \overline{x}, \overline{p})$ we get

$$v_{p_i} - \varphi_{p_i} \geq 0 \quad \text{for } i \in I, \qquad v_{p_i} - \varphi_{p_i} = 0 \quad \text{for } i \in \{1,\ldots,n\} \backslash I \tag{5.5}$$

at $(\bar{t}, \overline{x}, \overline{p})$. Since we assume that $\widehat{u}$ is admissible w.r.t. $(\overline{x}, \overline{p})$, then we have $\widehat{u}_i \geq 0$ for $i \in I$, and therefore by (5.5), at $(\bar{t}, \overline{x}, \overline{p})$

$$(v_p - \varphi_p)^\tau \widehat{u} \geq 0. \tag{5.6}$$

Now by (5.4) and (5.6) we see that

$$-\frac{\partial \varphi}{\partial t} + \left\{ -\frac{\partial^\tau \varphi}{\partial z} \psi \right\} \bigg|_{u = \widehat{u}} - \frac{1}{2} \mathrm{tr} \left(\frac{\partial^2 \varphi}{\partial z^2} G G^\tau \right) - l \geq 0,$$

and then condition (ii) holds at $(\bar{t}, \overline{x}, \overline{p})$.

As in section 2, we also define the set $\mathcal{U} = \{u(\cdot) | u$ is adapted to $\sigma(z_s,\ s \leq t)$ and $u(t) \in U,\ t \leq T\}$.

LEMMA 5.2. *For any initial pair (s_0, x_0, p_0) with each $(p_0)_i \geq 0$, and any $u \in \mathcal{U}$, there exists $\widetilde{u} \in \mathcal{U}^{p_0}$ such that*

$$P_\Omega \left\{ \int_{s_0}^T |\widetilde{u} - u| ds \leq 4\varepsilon \right\} = 1, \tag{5.7}$$

where with probability 1 *and for all* $1 \le i \le n$*, the constant* $\varepsilon > 0$ *satisfies*

$$\sup_{t\in[s_0,T]} \max\{-p_i(t, s_0, p_0, u), 0\} \le \varepsilon, \tag{5.8}$$

and $p(t, s_0, p_0, u)$ *denotes the value of* p *at* t *with initial condition* (s_0, p_0) *and control* u*.*

Proof. We need only to modify each component u_i of u in the following way. Define $\tau_i^0 = s_0$, and for $k \ge 1$,

$$\tau_i^k = \inf\{t > \tau_i^{k-1},\ p_i(t, s_0, p_0, \widetilde{u}) = 0\}, \tag{5.9}$$

$$\tau_i^k = T \quad \text{if} \ \ p_i(t, \tau_i^{k-1} + \varepsilon, p_i(\tau_i^{k-1} + \varepsilon), u) > 0 \quad \forall t \ge \tau_i^{k-1} + \varepsilon, \tag{5.10}$$

$$\widetilde{u}_i(t) = 1 \quad \text{on } [\tau_i^{k-1},\ \tau_i^{k-1} + \varepsilon), \tag{5.11}$$

$$\widetilde{u}_i(t) = u_i(t) \quad \text{on } [\tau_i^{k-1} + \varepsilon, \tau_i^k). \tag{5.12}$$

Then it is obvious that $\widetilde{u} \in \mathcal{U}^{p_0}$. Suppose that (5.7) is not true, and then there exist i and a set A^0 with $P_\Omega(A^0) > 0$ such that on A^0

$$\int_{s_0}^{T} |\widetilde{u}_i - u_i| ds > 4\varepsilon. \tag{5.13}$$

For any fixed $\omega \in A^0$, if $\tau_i^{k_0}$ is the last stopping time defined by (5.9) and (5.10), then by (5.13) we can easily show that $p_i(\tau_i^{k_0-1}, s_0, p_0, u) < -2\varepsilon$, which is a contradiction to (5.8). ∎

With Lemma 5.2, we can further show that the value function $v(t, z)$ is continuous on $\overline{Q}_T$ by a comparison method, as in the unconstrained case [3]. The details are omitted here. The growth condition of Proposition 2.2 also holds in the state constrained case.

PROPOSITION 5.3. *The value function* v *is a constrained viscosity solution to the HJB equation* (5.1)*.*

Proof. We verify condition (i) first. For an initial condition pair (s, z) with $z \in Q_T^0$ and any $u \in U$ we construct control $\widetilde{u} = u$ on $[s, s+\varepsilon]$ and $\widetilde{u} = 0$ on $(s+\varepsilon, T]$. We see that when ε is sufficiently small, $\widetilde{u}$ is in the admissible control set w.r.t. (s, z) since each $p_i \in [0, \infty)$. All the remaining steps and the verification of condition (ii) can be done as in Theorem 2.4. ∎

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