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Graphon Mean Field Games and the GMFG Equations: ε-Nash Equilibria

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***Abstract*— Very large networks linking dynamical agents are now ubiquitous and the need to analyse, design and control them is evident. The emergence of the graphon theory of large networks and their infinite limits has enabled the formulation of a theory of the centralized control of dynamical systems distributed on asymptotically infinite networks [Gao and Caines, CDC 2017, 2018]. Moreover, the study of the decentralized control of such systems was initiated in [Caines and Huang, CDC 2018] where Graphon Mean Field Games (GMFG) and the GMFG equations were formulated for the analysis of non-cooperative dynamic games on unbounded networks. In that work, existence and uniqueness results were established for the GMFG equations, while the current work continues that analysis by developing an ε-Nash theory for GMFG systems by relating the infinite population equilibria on infinite networks to finite population equilibria on finite networks.**

I. Introduction

A strategy to confront the problem of huge system complexity is to pass to an appropriately formulated infinite limit. This approach has a distinguished history since it is the conceptual principle underlying the celebrated Boltzmann Equation of statistical mechanics and that of the fundamental Navier-Stokes equation of fluid mechanics (see e.g. [1], [2], [3], [4]). Similarly the Fokker-Plank-Kolmogorov (FPK) equations for the macroscopic flow of probabilities [5], [6] is used to describe a vast range of phenomena which at the fine micro or mezzo level are modelled via the random interactions of discrete entities.

The work in this paper is formulated within two recent theories developed with an analogous motive to that alluded to above, namely Mean Field Game theory for the analysis of equilibria in very large populations of non-cooperative agents (see [7], [8], [9], [10], [11], [12], [13]), and the graphon theory of the infinite limits of graphs and networks (see [14], [15], [16], [17], [18]).

There have been mathematically rigorous studies of MFG systems with state values in finite graphs (see e.g. [19]), and of MFG systems where the agent subsystems are defined at the nodes (vertices) of finite random Erdös-Rényi graphs [20]. In that work the issue of system behaviour subject to the unbounded growth of the network is not analysed. However, graphon theory gives a rigorous formulation of the notion of limits for infinite sequences of networks of increasing size, and the first application of graphon theory in dynamics appears to be the work in [21], [22] and [23].

The first applications of graphon theory in systems and control theory are those in [24], [25], [26], [27], which treat the centralized and distributed control of arbitrarily large networks of dynamical control systems for which a direct solution would be completely intractable. Approximate control is achieved by solving control problems on the infinite limit graphon and then applying control laws derived from those solutions on the finite network of interest. The analogy with the strategies for finding feedback laws resulting in ε-Nash equilibria in the MFG framework is obvious. In this connection we note that initial work on static game theoretic equilibria for infinite populations on graphons was reported in [28].

It may be seen that a natural framework for the formulation of game theoretic problems involving agents distributed over large networks is given by Mean Field Game theory defined on graphons. The resulting basic idea and the associated fundamental equations for what we term graphon Mean Field Game (GMFG) systems and the GMFG equations are the subject of the current paper and its predecessor [29]. The GMFG equations are of great generality since they permit the study, in the limit, of both dense and sparse, infinite networks of non-cooperative dynamical agents. Moreover the classical MFG equations are retrieved when the communication over the infinite network (modelled as a graphon) involves uniform weightings of a direct influence of all agents on the network on every other agent on the network. (We observe that an early analysis of linear quadratic models can be found in [30] on the topic of non-uniform weightings in mean field games, but there is no topology on the set of systems or any use of graphon theory.)

The previous paper [29] established the existence and uniqueness of solutions to the GMFG equations under suitable conditions. In the current paper that analysis is continued with the development of an ε-Nash theory for GMFG systems by relating the infinite population equilibria on infinite networks to finite population equilibria on finite networks.

II. The Concept of a Graphon

The basic idea of the theory of graphons is that the edge structure of each finite cardinality network is represented by a step function density on the unit square in $\mathbb{R}^2$ and a so-called cut-metric is introduced. The set of finite graphs endowed with the cut metric then gives rise to a metric space, and the completion of this space is the space of *graphons*. Graphons

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are represented by bounded symmetric Lebesgue measurable functions $W : [0,1]^2 \to [0,1]$ which can be interpreted as weighted graphs on the vertex set $[0,1]$. Despite the fact that finite set valued functions W satisfy this definition (and so in particular W functions defined on finite graphs) we shall reserve the use of the term graphon to the infinite valued case.

To be specific, unless otherwise stated, the term "graphon" here is used to refer to measurable functions $W_1 : [0,1]^2 \to [-1,1]$ and $\mathbf{G_1^{sp}}$ denotes the space of graphons. Let $\mathbf{G^{sp}}$ denote the space of all symmetric measurable functions $W : [0,1]^2 \to \mathbb{R}$.

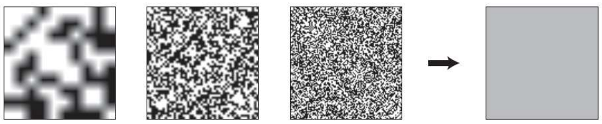

Fig. 1. E-R Graph Sequence Converging to Limit [18]

The cut norm of a graphon then has the expression:

$$\|W\|_{\square} = \sup_{M,T\subset[0,1]} \left| \int_{M\times T} W(x,y)dxdy \right| \qquad (1)$$

with the supremum taking over all measurable subsets M and T of $[0,1]$. Denote the set of measure preserving bijections $[0,1] \to [0,1]$ by $S_{[0,1]}$. The *cut metric* between two graphons V and W is then given by $d_{\square}(W,V) = \inf_{\phi\in S_{[0,1]}} \|W^{\phi} - V\|_{\square}$, where $W^{\phi}(x,y) := W(\phi(x),\phi(y))$.

The space $(\mathbf{G_1^{sp}}, d_{\square})$ is compact and this still holds if $\mathbf{G_1^{sp}}$ is replaced by any bounded subset of $\mathbf{G^{sp}}$ closed in the $d_{\square}$ distance [18]. Sets in $\mathbf{G_1^{sp}}$ compact with respect to the L^2 metric are compact with respect to the cut metric. It follows that if a graphon sequence is Cauchy in the L^2 metric then it is also a Cauchy sequence in the cut metric and the limits are identical in $\mathbf{G_1^{sp}}$.

In this paper, we start the modeling of the game of a finite population based on a finite graph. Specifically, the population resides on a weighted finite graph G_k with a set of nodes $\mathscr{V}_k = \{1,\ldots,M_k\}$ and weights $g_{ij}^k \in [0,1]$ for $(i,j) \in \mathscr{V}_k \times \mathscr{V}_k$. It is allowed to have $i = j$ in g_{ij}^k. We call $g_i^k := (g_{i1}^k, \cdots, g_{iM_k}^k)$ a section of g^k at i. Each node l is occupied by a set of players which is called a cluster of the population. So the number of clusters is M_k. Each cluster can be represented as the set of indices of the constituent agents. We list the clusters as $C_1,\ldots,C_{M_k}$. Without loss of generality, we assume the lth cluster occupies node l. Let $C(i)$ denote the cluster that agent i belongs to. So $i \in C(i)$. Our further analysis in the paper is based on the convergence of g^k to a graphon limit g. To indicate its arguments, we may write $g(\alpha,\beta)$ or alternatively $g_{\alpha,\beta}$. We define the section of g at α by $g_\alpha : \beta \mapsto g_{\alpha,\beta}$, $\beta \in [0,1]$.

Since clusters C_{i_1} and C_{i_2} reside on nodes i_1 and i_2 of G_k, respectively, we define $g_{C_{i_1}C_{i_2}}^k = g_{i_1,i_2}^k$. Similarly, we define the section $g_{C_i}^k = g_i^k$.

We partition $[0,1]$ into M_k subintervals of equal length. Here $I_l^k = [(l-1)/M_k, l/M_k]$ for $1 \le l \le M_k$. When it is clear from the context, we omit the superscript k and write I_l. To relate the clusters of agents to the vertex set $[0,1]$, we let the cluster C_l correspond to I_l.

III. Graphon MFG Systems and the MFG Equations

A. The Standard MFG Model and Its Graphon Generalization

In the diffusion based models of large population games the state evolution of a collection of N agents $\mathscr{A}_i, 1 \le i \le N < \infty$, is specified by a set of N controlled stochastic differential equations (SDEs). A simplified form of the general case is given by the following set of controlled SDEs which for each agent $\mathscr{A}_i$ includes state coupling with *all* other agents:

$$dx_i(t) = \frac{1}{N}\sum_{j=1}^{N} f(x_i(t),u_i(t),x_j(t))dt + \sigma dw_i(t), \qquad (2)$$

where $x_i \in \mathbb{R}^n$ is the state, $u_i \in \mathbb{R}^m$ the control input, and $w_i \in \mathbb{R}^r$ a standard Wiener process, and where $\{w_i, 1 \le i \le N\}$ are independent processes. For simplicity, all collections of system initial conditions are taken to be independent and have finite second moment. The cost is given by

$$J_i^N(u_i,u_{-i}) := E\int_0^T (1/N)\sum_{j=1}^{N} L(x_i(t),u_i(t),x_j(t))dt, \qquad (3)$$

where $1 \le i \le N$ and $L(\cdot)$ is the pairwise cost rate function, and u_{-i} denotes the controls of all agents other than $\mathscr{A}_i$.

The dynamics of a generic agent $\mathscr{A}_i$ in the infinite population limit of this system is then described by the controlled McKean-Vlasov (MV) equation

$$dx_i = f[x_i,u_i,\mu_t]dt + \sigma dw_i, \quad 0 \le t \le T, \qquad (4)$$

where $\mu_t(\cdot)$ denotes the distribution of the state of $\mathscr{A}_i$ in the population at $t \in [0,T]$, $f[x,u,\mu_t] := \int_{\mathbb{R}^n} f(x,u,y)\mu_t(dy)$ and where the initial condition measure μ_0 is specified. Setting $L[x,u,\mu_t] = \int_{\mathbb{R}^n} L(x,u,y)\mu_t(dy)$, the corresponding infinite population cost for $\mathscr{A}_i$ takes the form

$$J_i(u_i,\mu) \;:=\; E\int_0^T L[x_i(t),u_i(t),\mu_t]dt. \qquad (5)$$

Now we consider a finite population distributed over the finite graph G_k. Let $\mathbf{x}_{G_k} = \bigoplus_{l=1}^{M_k}\{x_i | i \in C_l\}$ denote the states of all agents in the total set of clusters of the population. This gives a total of $N = \sum_{l=1}^{M_k} |C_l|$ individual states.

For $\mathscr{A}_i$, the coupling term in the dynamics takes the form

$$f_{G_k}(x_i,u_i,g_{C(i)}^k) = \frac{1}{M_k}\sum_{l=1}^{M_k} g_{C(i)C_l}^k \frac{1}{|C_l|}\sum_{j\in C_l} f(x_i,u_i,x_j).$$

The specification of f_{G_k} relies on the sectional information $g_{C(i),\bullet}^k$ of $\mathscr{A}_i$. Concerning this coupling structure we observe that from the point of $\mathscr{A}_i$, all individuals residing in cluster C_l are symmetric and their average generates an overall impact of that cluster. Subsequently, all spatially distributed clusters form a weighed average according to g^k defined on the finite graph G_k. Denote

$$\tilde{f}_{G_k}(x_i,u_i,g_{C(i)}^k) = f_0(x_i,u_i) + f_{G_k}(x_i,u_i,g_{C(i)}^k).$$

The state process of $\mathscr{A}_i$ is given by the stochastic differential equation

$$dx_i(t) = \tilde{f}_{G_k}(x_i, u_i, g^k_{C(i)})dt + \sigma dw_i, \tag{6}$$

where the initial state is $x_i(0)$.

Without justification at this point, we introduce the following expression for the limit as the number of nodes of the graph G_k and each subpopulation located at each of its nodes tend to infinity:

$$f[x_\alpha, u_\alpha, \mu_G; g_\alpha] := \int_0^1 \int_{\mathbb{R}^n} f(x_\alpha, u_\alpha, z) g(\alpha, \beta) \mu_\beta(dz) d\beta,$$

which gives the complete local graphon dynamics via

$$\widetilde{f}[x_\alpha, u_\alpha, \mu_G; g_\alpha] := f_0(x_\alpha, u_\alpha) + f[x_\alpha, u_\alpha, \mu_G; g_\alpha]. \tag{7}$$

Due to the integration with respect to β, the dependence of $\widetilde{f}$ on g is through the section g_α.

Finally, parallel to the standard MFG case, the stochastic differential equation

$$\begin{aligned} &[\text{MV-SDE}](\alpha) \quad dx_\alpha = \widetilde{f}[x_\alpha, u_\alpha, \mu_G; g_\alpha]dt + \sigma dw_t, \\ &\quad 0 \le t \le T, \quad \alpha \in [0,1], \end{aligned} \tag{8}$$

corresponds in the graphon case to the standard controlled MV equation (4).

Analogously, in the Graphon Mean Field case, we define the coupled cost for $\mathscr{A}_i$ in the cluster $C(i)$ to be

$$L_{G_k}(x_i, u_i, g^k_{C(i)}) = \frac{1}{M_k} \sum_{l=1}^{M_k} g^k_{C(i)C_l} \frac{1}{|C_l|} \sum_{j \in C_l} L(x_i, u_i, x_j).$$

Define $\tilde{L}_{G_k}(x_i, u_i, g^k_{C(i)}) = L_0(x_i, u_i) + L_{G_k}(x_i, u_i, g^k_{C(i)})$. The cost of the agent is given in the form

$$J_i = E \int_0^T \tilde{L}_{G_k}(x_i, u_i, g^k_{C(i)})dt. \tag{9}$$

Denote

$$L[x_\alpha, u_\alpha, \mu_G; g_\alpha] = \int_0^1 \int_{\mathbb{R}^n} L(x_\alpha, u_\alpha, z) g(\alpha, \beta) \mu_\beta(dz) d\beta$$

and

$$\widetilde{L}[x_\alpha, u_\alpha, \mu_G; g_\alpha] = L_0(x_\alpha, u_\alpha) + L[x_\alpha, u_\alpha, \mu_G; g_\alpha].$$

In the infinite population graphon case, the individual agent α has the cost function given by

$$J_\alpha(u_\alpha, \mu_G) = E \int_0^T \widetilde{L}[x_\alpha(t), u_\alpha(t), \mu_G; g_\alpha]dt. \tag{10}$$

B. The Graphon MFG Model and Its Equations

In this section the standard MFG equations (see e.g. [31], [13]) will be generalized so that they subsume the standard (implicitly uniform totally connected) dense network case and cover the fully general graphon limit network case. Specifically, agent $\mathscr{A}_i$ in a population of N agents will be located at the lth node in an M_k node network and in the infinite population graphon limit that node will be taken to map to $\alpha \in [0,1]$. It is important to note here that *although the network is assumed dense it is not assumed to be uniformly totally connected*; indeed, the connection structure of the infinite network is represented precisely by its graphon $G = \{g(\alpha, \beta), 0 \le \alpha, \beta \le 1\}$.

The generalized Graphon MFG scheme below on $[0, T]$ is given by the linked equations for (i) the value function V_α for a generic agent's stochastic control problem when all other agents' control laws are fixed and generating the given local mean field μ_α and the graphon local mean field μ_β, (ii) the FPK for the MV-SDE for the local mean field of the generic agent, and (iii) the specification of the best response feedback law.

The key feature of the generalized graphon MFG construction beyond the standard MFG scheme is that at any agent in a dense network the averaged dynamics (2) and cost function (3) decompose into averages of neighbouring subpopulations distributed on the network edges incident upon that agent's node plus a standard local differential dynamics. In the limit, the summed subpopulation averages are given by an integral over the local mean field measures of the neighbouring agents. For notational simplicity, we present the graphon MFG framework with scalar individual states and controls, i.e., $n = m = r = 1$. Its extension to the vector case is evident. Specifically, (suppressing the time index on the measures for simplicity of notation) we have the *Graphon Mean Field Game (GMFG) equations*:

$$\begin{aligned} [\text{HJB}](\alpha) \quad -\frac{\partial V_\alpha(t,x)}{\partial t} &= \inf_{u \in U} \left\{ \widetilde{f}[x, u, \mu_G; g_\alpha] \frac{\partial V_\alpha(t,x)}{\partial x} \right. \\ &\left. + \widetilde{L}[x, u, \mu_G; g_\alpha] \right\} + \frac{\sigma^2}{2} \frac{\partial^2 V_\alpha(t,x)}{\partial x^2}, \end{aligned} \tag{11}$$

$$V_\alpha(T, x) = 0, \quad (t,x) \in [0,T] \times \mathbb{R}, \quad \alpha \in [0,1],$$

$$\begin{aligned} [\text{FPK}](\alpha) \quad \frac{\partial p_\alpha(t,x)}{\partial t} &= -\frac{\partial \{\widetilde{f}[x, u^0, \mu_G; g_\alpha] p_\alpha(t,x)\}}{\partial x} \\ &+ \frac{\sigma^2}{2} \frac{\partial^2 p_\alpha(t,x)}{\partial x^2}, \end{aligned} \tag{12}$$

$$[\text{BR}](\alpha) \quad u^0(x_\alpha, \mu_G; g_\alpha) =: \varphi(t, x_\alpha | \mu_G; g_\alpha)$$

Here the densities $p_\alpha(t,x)$ of the measures $\mu_\alpha \equiv \mu_\alpha(t)$ are assumed to exist, and where, to complete the specification of the GMFG equations, we define the following terms: the *graphon local mean field* μ_α and the corresponding set or ensemble of all the *local mean fields* $\mu_G = \{\mu_\beta, 0 \le \beta \le 1\}$, the *local dynamics* $f_0(x_\alpha, u_\alpha)$, the *network local dynamics* $f(x_\alpha, u_\alpha, x_\beta)$, and the *graphon averaged local dynamics* $f[x_\alpha, u_\alpha, \mu_G; g_\alpha]$ of the α indexed system and the graphon section function $g_\alpha = \{g(\alpha, \beta); 0 \le \beta \le 1\}$.

In a similar manner, $\widetilde{L}$ is defined based on two functions $L_0(x_\alpha, u_\alpha)$, $L(x_\alpha, u_\alpha, x_\beta)$ and g_α. Note that in (11) and (12), μ_G depends on time t and may be written as $\mu_G(t)$.

This completes the GMFG specification.

We notice that we retrieve the simplest standard MFG framework, where the agents' dynamics and costs are uniform, and where the network is totally connected with uniform link weights, by setting $\{g(\alpha, \beta) = 1, 0 \le \alpha, \beta \le 1\}$. Then at each node (by uniformity) the local MKV state distributions may be taken to be equal to that at a nominal node α, in other words $\mu_\beta(dx_\beta) = \mu_\alpha(dx_\beta)$ for all β. In that case, at the instant t, by (7), $\widetilde{f}[x,u,\mu_G(t);g_\alpha]$ takes the standard MKV dynamics form $f[x,u,\mu_\alpha(t)]$, and the local and graphon local measures are equal: $\mu_\alpha(t) =: \mu(t)$, for all α. Evidently this system wide mean field state distribution is given by the standard form of equation (12) (see [31], [13]).

It is to be noted that the GMFG best response control of each agent generalizes the fundamental property of a solution to the MFG equations in the standard case where for a generic agent *the feedback control (BR) generates a Nash equilibrium* with the best response strategy depending only on the agent's state, x_α, and the standard mean field, namely the generic agent's state distribution $\mu(t)$.

Finally we note that the standard case of controlled diffusion processes is simply obtained by setting $\{g(\alpha,\beta) = 0; 0 \le \alpha,\beta \le 1\}$, which totally disconnects the network and results simply in $\widetilde{f}[x,u,\mu_G(t);g_\alpha] = f_0(x,u)$, and $\widetilde{L}[x,u,\mu_G(t);g_\alpha] = L_0(x,u)$.

In order to analyze the solvability of the GMFG equations, we need to restrict $\mu_G(\cdot)$ to a certain class. We say $\{\mu_G(t), 0 \le t \le T\}$ is from the admissible set $\mathscr{M}_{[0,T]}$ if

M1) for each fixed t, $\int_B \mu_\beta(t,dy)$ is a Lebesgue measurable function of β, where B is a Lebesgue measurable set;

M2) there exists $\eta \in (0,1]$ such that for any bounded and Lipschitz continuous function ϕ on $\mathbb{R}$,

$$\sup_{\beta\in[0,1]} \Big| \int_{\mathbb{R}} \phi(y)\mu_\beta(t_1,dy) - \int_{\mathbb{R}} \phi(y)\mu_\beta(t_2,dy)\Big| \le C_h|t_1-t_2|^\eta$$

where C_h may be selected to depend only on the Lipschitz constant $\mathrm{Lip}(\phi)$ for ϕ.

Remark 1: Condition M1) ensures the integration with respect to $d\beta$ in (7) is well defined. By condition M2), the drift term in the HJB equation has a certain time continuity, which facilitates the existence analysis of the best response.

A collection of measures on some measurable space which are indexed by the vertex set $[0,1]$ is called a measure ensemble. Thus, for each fixed t, $\mu_G(t)$ is a measure ensemble.

C. Assumptions for the Existence Analysis

To make the paper more self-contained and easier to read, we summarize the existence analysis developed in [29].

We introduce the following assumptions:

(H1) U is a compact set.

(H2) $f(x,u,y)$ and $L(x,u,y)$ ($f_0(x,u)$ and $L_0(x,u)$, resp.) are continuous and bounded functions on $\mathbb{R}\times U\times\mathbb{R}$ ($\mathbb{R}\times U$, resp.), and are Lipschitz continuous in (x,y) (in x, resp.) uniformly with respect to u.

(H3) For f_0, f and L_0, L, their first and second derivatives with respect to x are all uniformly continuous and bounded in $\mathbb{R}\times U\times\mathbb{R}$ (or $\mathbb{R}\times U$).

(H4) $f(x,u,y)$ ($f_0(x,u)$, resp.) is Lipschitz continuous in u, uniformly with respect to (x,y) (to x, resp.).

(H5) For any $q\in\mathbb{R}$, $\alpha\in[0,1]$ and any probability measure ensemble μ_G satisfying M1), the set

$$S(x,q) = \arg\min_u \{q(\widetilde{f}[x,u,\mu_G;g_\alpha]) + \widetilde{L}[x,u,\mu_G;g_\alpha]\} \quad (13)$$

is a singleton, and the resulting u as a function of (x,q), is Lipschitz continuous in (x,q), uniformly with respect to μ_G and g_α.

Although the GMFG equation system only involves $\{\mu_G(t), 0 \le t \le T\}$, which may be viewed as a collection of marginals at different vertices, it is necessary to develop the existence analysis in the underlying probability spaces (see related discussions in [7, p.240]).

We begin by introducing some analytic preliminaries. For the space $C_T = C([0,T],\mathbb{R})$, we specify a σ-algebra $\mathscr{F}_T$ induced by all cylindrical sets of the form $\{x(\cdot)\in C_T : x(t_i)\in B_i, 1\le i\le l \text{ for some } l\}$, where B_i is a Borel set. Let M_T denote the space of all probability measures on $(C_T,\mathscr{F}_T)$. The canonical process X is defined by $X_t(\omega) = \omega_t$ for $\omega\in C_T$. On C_T, we introduce the metric $\rho(x,y) = \sup_t |x(t)-y(t)|\wedge 1$. Then (C_T,ρ) is a complete metric space. Based on ρ, we introduce the Wasserstein metric on M_T. For $m_1, m_2\in M_T$, denote

$$D_T(m_1,m_2) = \inf_m \int_{C_T\times C_T} (\sup_{s\le T} |X_s(\omega_1) - X_s(\omega_2)|\wedge 1)\, dm(\omega_1,\omega_2),$$

where m is called a coupling as a probability measure on $(C_T,\mathscr{F}_T)\times(C_T,\mathscr{F}_T)$ with the first and second marginals as m_1 and m_2. Then (M_T,D_T) is a complete metric space [32].

We introduce the following product of probability measure spaces $\prod_{\alpha\in[0,1]}(C_T,\mathscr{F}_T,m_\alpha)$, where each individual space is interpreted as the path space of the agent at vertex α with a corresponding probability measure m_α. Denote the product of spaces of probability measures $M_{T,G} = \prod_{\alpha\in[0,1]} M_T$. An element in $M_{T,G}$ is a measure ensemble. Given a measure ensemble m_G, the projection operator Proj_α picks up its component associated with $\alpha\in[0,1]$.

For two measure ensembles $m_G := (m_\alpha)_{\alpha\in[0,1]}$ and $\bar{m}_G := (\bar{m}_\alpha)_{\alpha\in[0,1]}$ in $M_{T,G}$, define $d(m_G,\bar{m}_G) = \sup_{\alpha\in[0,1]} D_T(m_\alpha,\bar{m}_\alpha)$. Then $(M_{T,G},d)$ is a *complete metric space* [29].

Given the probability measure m_α, we determine the t-marginal as follows: $\mu_\alpha(t,B) = m_\alpha(\{x(\cdot)\in C_T : x(t)\in B\})$ for a Borel set $B\subset\mathbb{R}$, which will simply be denoted by $\mu_\alpha(t)$. Consider $m_G\in M_{T,G}$ and denote the time t marginal on a measure ensemble by the following rule

$$\mu_G(t) := (\mu_\alpha(t))_{\alpha\in[0,1]} = \mathrm{Marj}_t(m_G). \quad (14)$$

For a given t, this can be interpreted as a measure valued function defined on the vertex set $[0,1]$.

We introduce the sensitivity condition (see [29] for details):

$$\sup_{t,x,\alpha} |\phi_\alpha(t,x|\mu_G) - \bar{\phi}_\alpha(t,x|\bar{\mu}_G)| \le c_1 D_T(m_G,\bar{m}_G). \quad (15)$$

The set of control laws $\{\phi_\alpha(t,x|\mu_G(\cdot)), \alpha\in[0,1]\}$ (resp., $\{\bar{\phi}_\alpha(t,x|\bar{\mu}_G(\cdot)), \alpha\in[0,1]\}$) is determined by use of μ_G (resp., $\bar{\mu}_G$) in the optimal control problem (8) and (10) with the graphon section g_α. This is a generalization from the finite class model in [7] where an illustration via a linear model is presented.

As in [7, Lemma 9], we can show at each iteration to generate the individual process from the control law,

$$D_T(m_\alpha^{\text{new}}, \overline{m}_\alpha^{\text{new}}) \le c_2 \sup_{t,x} |\phi_\alpha(t,x|\mu_G(\cdot)) - \bar{\phi}_\alpha(t,x|\overline{\mu}_G(\cdot))|$$

for some constant c_2 not depending on α. Here m_α^{new} is the probability measure on the path space of the player at α vertex when all players apply the set of strategies $\{\phi_\alpha(t,x|\mu_G(\cdot)), \alpha \in [0,1]\}$. Similarly, $\overline{m}_\alpha^{\text{new}}$ corresponds to $\{\phi_\alpha(t,x|\overline{\mu}_G(\cdot)), \alpha \in [0,1]\}$.

D. Existence Theorem

We state the main result on the existence and uniqueness of solutions to the GMFG equation system. We introduce a contraction condition:

(H6) $c_1 c_2 < 1$.

Theorem 1: [29] Under (H1)-(H6), there exists a unique solution to the GMFG equations, which (i) gives the feedback control best response (BR) strategy $\varphi(t,x_\alpha|\mu_G;g_\alpha)$ depending only upon the agent's state and the graphon local mean fields (i.e. $(x_\alpha, \mu_G; g_\alpha)$), and (ii) generates a Nash equilibrium. ■

Remark 2: By SDE estimates, one can obtain refined bound information on c_2. When the network coupling effect is weak, a small value for c_2 can be obtained.

Remark 3: For linear models, a verification of the contraction condition can be done under certain model parameters, as in [7].

IV. The Performance Analysis

In the MFG case it is shown [7], [13] that the joint strategy $\{u_i^o(t) = \varphi_i(t,x_i(t)|\mu_t), 1 \le i \le N\}$ yields an ε-Nash equilibrium for all ε, i.e. for all $\varepsilon > 0$, there exists $N(\varepsilon)$ such that for all $N \ge N(\varepsilon)$

$$J_i^N(u_i^\circ, u_{-i}^\circ) - \varepsilon \le \inf_{u_i \in \mathscr{U}_i} J_i^N(u_i, u_{-i}^\circ) \le J_i^N(u_i^\circ, u_{-i}^\circ). \qquad (16)$$

This form of approximate Nash equilibrium is in fact a principal result of all the MFG work in the contributions in the sequence [7], [13] to [33] and many other works in the literature. That is to say *the cost function of any agent in a finite population can be reduced (i.e. improved) by at most ε if it changes unilaterally from the infinite MFG population feedback law* while all other agents remain with the infinite population based control strategies.

This basic ε-Nash equilibrium result in MFG theory and its expected form in GMFG theory are not only theoretically significant but are vital for the application of MFG derived control laws since the solution of the MFG and GMFG equations is necessarily simpler than the effectively intractable task of finding the solution to the game problems for the large finite population systems. Indeed, this was one of the original motives for the creation of MFG theory to tackle complexity. Furthermore it is a basic feature of graphon systems control theory [24].

A. The ε-Nash Equilibrium

For the graphon MFG analyzed in this paper there is a double limit as stated by the following assumption.

(H7) We have $M_k \to \infty$ and $\min_{1 \le l \le M_k} |C_l| \to \infty$ as $k \to \infty$.

We introduce some further assumptions. (H8) below is a continuity assumption on the graphon function $g(\alpha,\beta)$.

(H8) For any bounded and measurable function $h(\beta)$, the function $\int_0^1 g(\alpha,\beta)h(\beta)d\beta$ is continuous in $\alpha \in [0,1]$.

(H9) The best response $\varphi(t,x|\mu_G;g_\alpha)$ as a bounded and continuous function of (t,x) depends continuously on $\alpha \in [0,1]$ (by the norm of $C([0,T]\times\mathbb{R};U)$), where x stands for the state of the α-agent.

(H10) All agents have i.i.d. initial states with distribution $\mu_0(dx)$ and $E|x_i(0)| \le C_0$.

Remark 4: (H10) is a simplifying assumption to keep further notation light. It may be generalized to α dependent initial distributions.

(H11) We have

$$\lim_{k\to\infty} \max_i \sum_{j=1}^{M_k} \Big| \frac{1}{M_k} g_{C_i,C_j}^k - \int_{\beta \in I_j} g_{I_i^*,\beta} d\beta \Big| = 0,$$

where I_i^* is the midpoint of $I_i \in \{I_1 \cdots I_{M_k}\}$ of length $1/M_k$.

Remark 5: Assumption (H11) characterizes the approximation error between g^k for the finite graph and the graphon function g.

For the ε-Nash equilibrium analysis, we consider a sequence of games each defined on a finite graph G_k. Recall that there is a total of $N = \sum_{l=1}^{M_k} |C_l|$ players.

Suppose the cluster $C(i)$ of agent $\mathscr{A}_i$ corresponds to the subinterval $I(i) \in \{I_1, \cdots, I_{M_k}\}$. The agent $\mathscr{A}_i$ takes the midpoint $I^*(i)$ of the sub-interval $I(i)$ and use the GMFG equations to determine its control law

$$\hat{u}_i = \varphi(t,x_i|\mu_G;g_{I^*(i)}), \quad 1 \le i \le N, \qquad (17)$$

which we simply write as $\varphi = \phi(t,x_i,g_{I^*(i)})$. Denote the resulting state process by $\hat{x}_i$, $1 \le i \le N$. Recall that

$$f_{G_k}(x_i^N, u_i^N, g_{C(i)}^k) = \frac{1}{M_k}\sum_{l=1}^{M_k} g_{C(i)C_l}^k \frac{1}{|C_l|}\sum_{j\in C_l} f(x_i^N, u_i^N, x_j^N),$$

where we add the superscript N to indicate the population size. So the closed-loop state processes are given by

$$\begin{aligned} d\hat{x}_i^N =& f_0(\hat{x}_i^N, \phi(t,\hat{x}_i^N, g_{I^*(i)}))dt \\ &+ f_{G_k}(\hat{x}_i^N, \phi(t,\hat{x}_i^N, g_{I^*(i)}), g_{C(i)}^k)dt + \sigma dw_i, \end{aligned} \qquad (18)$$

where $1 \le i \le N$ and $\hat{x}_i^N(0) = x_i^N(0)$. Note that $g_{C(i)}^k$ appears in the second term as determined by the finite population system dynamics. We state the following main result.

Theorem 2: (**ε-Nash equilibrium**) Assume (H1)-(H11). When the strategies (17) determined by the GMFG equations are applied to a finite graph model G_k, the ε-Nash equilibrium property holds where $\varepsilon \to 0$ as $k \to \infty$, and where the deviant player i uses centralized Lipschitz feedback strategies $\varphi(t,x_i,x_{-i})$.

Proof: We first explain the basic idea of showing the ε-Nash equilibrium property. Suppose all other players, except agent $\mathscr{A}_i$, have employed the control strategies based on the GMFG equation system. When $\mathscr{A}_i$ attempts to use a different strategy, the performance can be measured according to a limiting stochastic control problem where both the dynamics and the cost are subject to a small perturbation caused by the mean field approximation for the effect of all other players. The proof is long and we only give a sketch due to space limit. We split the proof into several steps.

Step 1. N independent processes for approximating $\hat{x}_1^N,\cdots,\hat{x}_N^N$. For the approximation of the system of N coupled agents in (18), we introduce the following system

$$\begin{aligned}dy_i^N =&f_0(y_i^N,\phi(t,y_i^N,g_{I^*(i)}))dt+\\ &\frac{1}{M_k}\sum_{l=1}^{M_k} g^k_{C(i)C_l}\frac{1}{|C_l|}\sum_{j\in C_l}\int_{\mathbb{R}} f(y_i^N,\phi(t,y_i^N,g_{I^*(i)}),z)m_{y_j^N}(dz)dt\\ &+\sigma dw_i\\ =&f_0(y_i^N,\phi(t,y_i^N,g_{I^*(i)}))dt\\ &+\frac{1}{M_k}\sum_{l=1}^{M_k} g^k_{C(i)C_l}\int f(y_i^N,\phi(t,y_i^N,g_{I^*(i)}),z)m_l^N(dz)dt\\ &+\sigma dw_i, \end{aligned}\tag{19}$$

where $1\le i\le N$ and $y_i^N(0)=x_i^N(0)$. The second equality holds since all processes in cluster C_l have the same distribution denoted by $m_l^N(dz)$ at time t. Each Brownian motion w_i is the same as in (18). It is clear that the processes $y_1^N,\cdots,y_N^N$ are independent. If two distinct indices i,j are in C_l, then $x_i^N(0)$ and $x_j^N(0)$ are i.i.d. So $y_i^N(t)$ and $y_j^N(t)$ are i.i.d.

Lemma 3: The SDE system (19) has a unique solution $(y_1^N,\cdots,y_N^N)$.

Proof. The proof is similar to [7]. ∎

Denote $\varepsilon_{1,N}=\sup_{i,t}E|\hat{x}_i^N(t)-y_i^N(t)|$.

Lemma 4: We have $\varepsilon_{1,N}\to 0$ as $N\to\infty$ ∎

Step 2. N processes generated by the GMFG system. Now we define the process

$$dy_i^\infty=\widetilde{f}[y_i^\infty,\phi(t,y_i^\infty,g_{I^*(i)}),\mu_G;g_{I^*(i)}\ ]dt+\sigma dw_i,\tag{20}$$

where $1\le i\le N$ and $y_i^\infty(0)=x_i^N(0)$. Here w_i is the same as in (18). This system is analogous to (8) and the control law is selected by (17). Note that if $j\in C(i)$, y_i^∞ and y_j^∞ are two processes of the same distribution.

Denote $\varepsilon_{2,N}=\sup_{i,t}E|y_i^N-y_i^\infty|$.

Lemma 5: We have $\varepsilon_{2,N}\to 0$ as $N\to\infty$. ∎

Lemma 6: We have $\lim_{N\to\infty}\sup_{t,i}E|\hat{x}_i^N-y_i^\infty|=0$.

Proof. The lemma follows from Lemmas 4 and 5. ∎

Step 3. One agent applying a different strategy. Suppose x_i^N is determined from u_i^N not as the GMFG best response. All other agents $\mathscr{A}_j$, $j\ne i$, have strategies determined by (17). We introduce the new system:

$$dx_i^N=f_0(x_i^N,u_i^N)dt+f_{G_k}(x_i^N,u_i^N,g^k_{C(i)})dt+\sigma dw_i,\tag{21}$$

and for $j\neq i$,

$$\begin{aligned}dx_j^N =&f_0(x_j^N,\phi(t,x_j^N,g_{I^*(j)}))dt\\ &+f_{G_k}(x_j^N,\phi(t,x_j^N,g_{I^*(j)}),g^k_{C(j)})dt+\sigma dw_j.\end{aligned}\tag{22}$$

We note that x_j^N is affected by x_i^N due to the coupling term f_{G_k}. For this reason, x_j^N differs from $\hat{x}_j^N$ although the control law of $\mathscr{A}_j$ is the same. The key observation is that no matter what u_i is chosen, its perturbation to other agents is small. Denote $\varepsilon_{3,N}=\sup_{u_i,t,j\ne i}E|x_j^N(t)-y_j^\infty(t)|$.

Lemma 7: We have $\lim_{N\to\infty}\varepsilon_{3,N}=0$.

Proof. We first show $E\sup_{u_i,t,j\neq i}|x_j^N-\hat{x}_j^N|=0$ by basic SDE estimates and next use Lemma 6. ∎

Step 4. Consider control laws as in Step 3. Denote

$$\delta_f=\sup_{z\in\mathbb{R},u\in U}|f_{G_k}(z,u,g^k_{C(i)})-f[z,u,\mu_G;g_{I^*(i)}]|,$$

where f_{G_k} is random depending on $(x_1^N,\cdots,x_N^N)$ in (21)-(22). By using the results in Step 3 and properties of f, we obtain an upper bound of $E\delta_f$, which tends to zero as $N\to\infty$. A similar error bound is obtained for

$$\delta_l=\sup_{z\in\mathbb{R},u\in U}|L_{G_k}(z,u,g^k_{C(i)})-L[z,u,\mu_G;g_{I^*(i)}]|.$$

Step 5. Finally, to establish the ε-Nash equilibrium, the cost of agent $\mathscr{A}_i$ within the N agents can be written using the mean field limit dynamics and cost, both involving μ_G, up to a small error term that can be bounded uniformly with respect to u_i as in Step 4, while agent $\mathscr{A}_i$ chooses its control u_i. It can further have little improvement due to the best response property of $\varphi(t,x_i|\mu_G,g_{I^*(i)})$. ∎

V. The LQ Case

This section considers a special class of linear-quadratic-Gaussian (LQG) GMFG models. Consider the graph G_k with vertices $\mathscr{V}_k=\{1,\cdots,M_k\}$ and graph adjacency matrix $g^k=[g^k_{jl}]$. For agent $\mathscr{A}_i$ in subpopulation cluster C_q situated in node q, let the graph averaged mean value of the system state at node q be denoted by z_i, where

$$z_i=\frac{1}{|M_k|}\sum_{l\in\mathscr{V}_k}g^k_{ql}\frac{1}{|C_l|}\sum_{j\in C_l}x_j,\quad x_i,\ z_i\in\mathbb{R}^n.$$

The dynamics of $\mathscr{A}_i$ are given by the linear system

$$dx_i=(Ax_i+Dz_i+Bu_i)dt+\Sigma dw_i,\quad 1\le i\le N,$$

where $u_i\in\mathbb{R}^m$ is the control input, $w_i\in\mathbb{R}^r$ is a standard Wiener process, and A,B,D,Σ are conformally dimensioned matrices.

The individual agent's cost function takes the form

$$\begin{aligned}J_i(u_i,\nu_i)=&E\int_0^T\big[(x_i-\nu_i)^TQ(x_i-\nu_i)+u_i^TRu_i\big]dt\\ &+E\big[(x_i(T)-\nu_i(T))^TQ_T(x_i(T)-\nu_i(T))\big],\quad 1\le i\le N,\end{aligned}$$

where $Q,\ Q_T\ge 0$, $R>0$, and $\nu_i=\gamma(z_i+\eta)$ is the process tracked by $\mathscr{A}_i$. Here $\eta\in\mathbb{R}^n$ and $\gamma\in\mathbb{R}$.

In the above LQG-GMFG system, the agents are coupled via their dynamics and cost functions over a finite bidirectional graph of clusters. Furthermore, the tracked process ν_i is stochastic since it depends on other agents' states.

In the infinite population and graphon limit case, the mean field coupling at α-agent (i.e., an agent situated at the α-vertex in $[0,1]$) is given by

$$z_\alpha = \int_{[0,1]} [g(\alpha,\beta) \int_{\mathbb{R}^n} x\mu_\beta(dx)]d\beta, \quad \alpha,\beta \in [0,1].$$

The individual agent's dynamics are given by

$$dx_\alpha = (Ax_\alpha + Dz_\alpha + Bu_\alpha)dt + \Sigma dw_\alpha, \quad \alpha \in [0,1].$$

The individual agent's cost function is

$$\begin{aligned} J_\alpha(u_\alpha, \nu_\alpha) = E\int_0^T & \left[(x_\alpha - \nu_\alpha)^T Q(x_\alpha - \nu_\alpha) + u_\alpha^T R u_\alpha\right]dt \\ & + E\left[(x_\alpha(T) - \nu_\alpha(T))^T Q_T (x_\alpha(T) - \nu_\alpha(T))\right], \end{aligned}$$

where $\nu_\alpha = \gamma(z_\alpha + \eta)$.

Denote the Riccati equation

$$\begin{aligned} &-\dot{\Pi}_t = A^T \Pi_t + \Pi_t A - \Pi_t B R^{-1} B^T \Pi_t + Q, \\ &-\dot{s}_\alpha(t) = \left(A - BR^{-1}B^T\Pi_t\right)^T s_\alpha(t) + \Pi_t D z_\alpha(t) - Q\nu_\alpha(t), \end{aligned}$$

where $\Pi_T = Q_T$ and $s_\alpha(T) = -Q_T\nu_\alpha(T)$. The optimal tracking control (as the best response) for an α-agent is given by

$$u_\alpha(t) = -R^{-1}B^T[\Pi_t x_\alpha(t) + s_\alpha(t)].$$

Here the graphon local mean field and tracked process from cost coupling are

$$z_\alpha = \int_{[0,1]} g(\alpha,\beta)\bar{x}_\beta d\beta, \quad \nu_\alpha = \gamma(z_\alpha + \eta), \quad \alpha \in [0,1],$$

where $\bar{x}_\beta = \int_{\mathbb{R}^n} x\mu_\beta(dx)$. The mean state process of x_α is

$$\dot{\bar{x}}_\alpha = (A - BR^{-1}B^T\Pi_t)\bar{x}_\alpha + Dz_\alpha - BR^{-1}B^T s_\alpha, \quad \alpha \in [0,1].$$

As an example for illustration, we assume the graphon local mean field at α-agent arises from an underlying *uniform attachment graphon*, and consequently

$$z_\alpha = \int_{[0,1]} \left[(1 - \max(\alpha,\beta)) \int_{\mathbb{R}^n} x\mu_\beta(dx)\right]d\beta,$$

where $\alpha,\beta \in [0,1]$.

VI. Conclusion

For future work, it is of great interest to develop computational techniques for GMFG problems.

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