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Brief paper

Mean field production output control with sticky prices: Nash and social solutions[☆]

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ABSTRACT

This paper presents an application of mean field control to dynamic production optimization with sticky prices and adjustment costs. Both noncooperative and cooperative solutions are considered. By solving auxiliary limiting optimal control problems subject to consistent mean field approximations, two sets of decentralized strategies are obtained and further shown to asymptotically attain Nash equilibria and social optima, respectively. A numerical example is given to compare market prices, firms' outputs and costs under the two solution frameworks.

1. Introduction

Mean field game theory is effective to design decentralized strategies in a system of many players which are individually negligible but collectively affect a particular player (see e.g., Huang, Caines, & Malhamé, 2007; Lasry & Lions, 2007). It combines mean field approximations and individual's best response to overcome the dimensionality difficulty. By now, mean field games have been intensively studied in the linear–quadratic–Gaussian (LQG) framework (Huang et al., 2007; Fallah, Malhame, & Martinelli, 2016; Huang & Huang, 2015). For further literature, readers are referred to Bensoussan, Frehse, and Yam (2013) and Li and Zhang (2008) for nonlinear models, Huang (2010) and Wang and Zhang (2012) for mean field models with a major player, and Weintraub, Benkard, and Van Roy (2008) for oblivious equilibria of industry dynamics. For a survey on mean field game theory, see Bensoussan et al. (2013) and Caines, Huang, and Malhame (2017). Besides noncooperative games, social optima in mean field control have been investigated in Huang, Caines, and Malhamé (2012) and Wang and Zhang (2017). Mean field control has found wide applications, including smart grids (Chen, Busic, & Meyn, 2015; Ma, Callaway, & Hiskens, 2013), finance, economics (Chan & Sircar, 2015; Guéant, Lasry, & Lions, 2011; Huang & Nguyen, 2016; Weintraub et al., 2008), operations research (Lucas & Moll, 2014), and social sciences (Bauso, Tembine, & Basar, 2016).

This paper aims to present an application of mean field control to production output adjustment in a large market with many firms and sticky prices, where the price of the underlying product does not adjust instantaneously according to its demand function but evolves slowly and smoothly. Dynamic game models for duopolistic competition with sticky prices were initially proposed by Simaan and Takayama (1978), and extended to investigate asymptotically stable steady-state equilibrium prices in Fershtman and Kamien (1987). In Cellini and Lambertini (2004) and Wiszniewska-Matyszkiel, Bodnar, and Mirota (2015), the authors considered open and closed-loop Nash equilibria for dynamic oligopoly with N firms and compared prices' behavior in and outside the steady-state levels, respectively. Production adjustment costs have been addressed in the economic literature (see e.g. Hall, 2002) and have been taken into account in the study of dynamic oligopoly (Driskill & McCafferty, 1989; Jun & Vives, 2004; Schoonbeek, 1997). The work (Driskill & McCafferty, 1989) introduces a duopoly where each firm has output level subject to control according to a first-order integrator dynamics. However, when the number of firms is large (e.g. in a perfectly competitive market) and the adjustment cost is considered, the computational complexity of output adjustment is high.

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Within our model, a large number of producers supply a certain product with sticky prices, and the output adjustment incurs a cost. The cost function of a firm is based on product cost, price, and adjustment cost. In Wang and Huang (2015), we combined the price and firm's output as a 2-dimensional system. This leads to a cost with indefinite state weights, which differs from many existing mean field LQG models in the literature (Huang et al., 2007, 2012; Wang & Zhang, 2017). In this paper, the price in the limiting model is taken as an exogenous signal without the need of state space augmentation. This contributes to deriving a simple condition for solvability of the resulting equation system.

Mean field control provides an ideal framework to address complexity. In this paper, we design Nash and social optimum strategies, respectively, for the production model based on the mean field control methodology, and further compare the two solutions numerically. Our main contributions are summarized as follows: (i) We propose a mean field LQG model for production output adjustment with sticky prices; (ii) Nash and social solutions are designed for the mean field model with indefinite state weights in costs; (iii) the performance estimate of the social optimum strategies exploits a passivity property of the system.

An illustrative example is given to compare market prices, firm's outputs and optimal costs under the game and social optimum frameworks. It is numerically shown that the social optimum has a lower average output than in the noncooperative case. This is similar to the behavior in a duopoly model (Varian, 1993) where collusion of two firms results in a lower total output than in the Cournot equilibrium.

The paper is organized as follows. Section 2 introduces the game and social optimum problems with N players. In Section 3, we design a set of decentralized strategies by the mean field control methodology and show its asymptotic Nash equilibrium property. In Section 4, we construct a set of socially optimal decentralized strategies. Section 5 compares the two solutions by a numerical example. Section 6 concludes the paper.

Notation: $\|\cdot\|$ denotes the Euclidean vector norm or matrix spectral norm. $C([0,\infty),\mathbb{R}^n)$ denotes the set of n-dimensional continuous functions on $[0,\infty)$; $C_b([0,\infty),\mathbb{R}^n)$ is the set of bounded and continuous functions; $C_\rho([0,\infty),\mathbb{R}^n) = \{f | f \in C([0,\infty),\mathbb{R}^n), \sup_{t\geq 0}|f(t)|e^{-\rho' t} < \infty$ for some $\rho' \in [0,\rho)\}$. For a family of $\mathbb{R}^n$-values random variables $\{x(\lambda), \lambda \geq 0\}$, $\sigma(x(\lambda), \lambda \leq t)$ is the σ-algebra generated by these random variables; $\|x\|_\rho = \left[E\int_0^\infty e^{-\rho t}|x(t)|^2 dt\right]^{1/2}$; $\|x\|_\infty = \sup_{t\geq 0}|x(t)|$. For convenience of presentation, we use $C, C_1, C_2, \ldots$ to denote generic positive constants, which may vary from place to place.

2. Problem description

2.1. Output adjustment in a mean field framework

According to the model in Cellini and Lambertini (2004) and Wiszniewska-Matyszkiel et al. (2015), the sticky price in dynamic oligopoly evolves by

$$\frac{dp}{dt} = \alpha(\beta - \delta\sum_{j=1}^{N} q_j - p), \quad p(0) \text{ given},$$

where q_j is the output of firm j, $j = 1,\ldots,N$, and has the role of control. The payoff function of firm i is described by

$$K_i(q_1,\ldots,q_N) = \int_0^\infty e^{-\rho t}(pq_i - cq_i - \tfrac{1}{2}q_i^2)dt.$$

The constants α, β, δ and c are positive, and c is the cost of unit output.

This paper considers a perfectly competitive market with many firms (Varian, 1993). Based on Fershtman & Kamien (1987) and Wiszniewska-Matyszkiel et al. (2015), we assume that the sticky price evolves by

$$\frac{dp(t)}{dt} = -\alpha p(t) - \alpha q^{(N)}(t) + \alpha\beta, \tag{1}$$

where $\alpha > 0$ denotes the speed of adjustment to the level on the demand function, and $q^{(N)}(t) = \frac{1}{N}\sum_{i=1}^{N} q_i(t)$ is the average output. The output of each firm is described by the stochastic differential equation (SDE)

$$dq_i(t) = -\mu q_i(t)dt + b_i u_i(t)dt + \sigma dw_i(t), \tag{2}$$

where $\{w_i(t), i = 1,\ldots,N\}$ are independent standard Brownian motions, which are also independent of $\{q_i(0), i = 1,\ldots,N\}$. The constants α, β, μ and b_i are positive.

Remark 1. As in Fershtman and Kamien (1987), $\beta - q^{(N)}$ is the price on the demand function for the given level of firms' outputs. In the static case, the inverse demand function has a linear version $p = \beta - \delta q^{(N)}$; here for simplicity we set δ as 1. The scaling factor $1/N$ for $q^{(N)}$ is standard in modeling large markets (Lambson, 1984; Ushio, 1989). μ is used to indicate friction in adjusting the output, and w_i is random shocks in output.

The cost function of each firm is given by

$$J_i(u) = E\int_0^\infty e^{-\rho t} L(p, q_i, u_i)dt, \tag{3}$$

where

$$L = -p(t)q_i(t) + cq_i(t) + ru_i^2(t),$$

$u = (u_1,\ldots,u_i,\ldots,u_N)$, $r > 0$ and $0 < c < \beta$. Here c denotes the production cost, and $ru_i^2(t)$ the adjustment cost. The minimization of $J_i(u)$ is equivalent to maximizing the payoff

$$K_i(u) = E\int_0^\infty e^{-\rho t}[q_i(t)(p(t) - c) - ru_i^2(t)]dt.$$

We only consider the case $\beta > c$ to make the subsequent optimization problems be of practical interest. Otherwise, given a positive $q^{(N)}$, the production cost already exceeds the price. The social cost is defined as

$$J_{\rm soc}^{(N)}(u) = \frac{1}{N}\sum_{j=1}^{N} J_j(u). \tag{4}$$

Based on costs (3) and (4), one may formulate a standard LQG game and an optimal control problem, respectively. A limitation of this approach is that the control strategy will be centralized.

The basic objective of this paper is to study the following two problems:

Problem I: Find ε-Nash equilibrium strategies for the agents to minimize the individual cost J_i over the set of decentralized strategies

$$\mathscr{U}_{d,i} = \left\{u_i : u_i(t) \text{ is adapted to } \mathscr{F}_t^i,\ E\int_0^\infty e^{-\rho t}u_i^2(t)dt < \infty\right\},$$

where $\mathscr{F}_t^i = \sigma\{q_i(0), w_i(s), s \leq t\}$, $t \geq 0$, $i = 1,\ldots,N$.

Problem II: Find asymptotic social optimum strategies for the agents to minimize $J_{\rm soc}^{(N)}$ over the set of strategies $\mathscr{U}_{d,i}$, $i = 1,\ldots,N$.

For a large market, a natural way of modeling the sequence of parameters $b_1,\ldots,b_N$ is to view them as being sampled from a space such that this sequence exhibits certain statistical properties when $N \to \infty$. Define the associated empirical distribution function $F_N(b) = \frac{1}{N}\sum_{i=1}^{N} I_{[b_i \leq b]}$, where $I_{[b_i\leq b]} = 1$ if $b_i \leq b$ and $I_{[b_i\leq b]} = 0$ otherwise.

We introduce the assumptions.

(A1) The initial price $p(0) = p_0 > 0$ is a constant. The initial outputs $\{q_i(0), i = 1,\ldots,N\}$ are independent. $Eq_i(0) = q_0 > 0$ for all $i = 1, \ldots, N$; there exists $c_0 < \infty$ independent of N such that $\max_{i=1,\ldots,N} E|q_i(0)|^2 \le c_0$.

(A2) There exists a distribution function F such that F_N converges weakly to F. Furthermore, each $b_i > 0$ and $\int_{\mathbb{R}} \theta^2 dF(\theta) > 0$.

(A3) For all N, $\{b_i, i = 1, \ldots, N\}$ is contained in a fixed compact set Θ, and $\int_\Theta \theta dF(\theta) = 1$.

3. Nash solutions to output adjustment

3.1. Optimal control for the limiting problem

Assume that $\bar{q} \in C_{\rho/2}([0,\infty), \mathbb{R})$ is given for approximation of $q^{(N)}$. Replacing $q^{(N)}$ in (1) by $\bar{q}$, we introduce

$$\frac{d\bar{p}(t)}{dt} = \alpha[\beta - \bar{p}(t) - \bar{q}(t)], \quad \bar{p}(0) = p_0. \tag{5}$$

After replacing p in (3) by $\bar{p}$, we define the cost function:

$$\bar{J}_i(u_i) = E\int_0^\infty e^{-\rho t}[(c - \bar{p})q_i + ru_i^2]dt. \tag{6}$$

The corresponding admissible control set is $\mathscr{U}_{d,i}$.

We first take $\bar{p}$ as an exogenous signal and solve the problem in (2), (5) and (6). For a general initial condition $q_i(t) = q_i$ at time t, define the value function

$$V_i(t, q_i) = \inf_{u_i \in \mathscr{U}_{d,i}} E\left[\int_t^\infty e^{-\rho(\tau - t)} L(\bar{p}, q_i, u_i)d\tau \middle| q_i(t) = q_i\right].$$

We introduce the HJB equation:

$$\rho V_i = \inf_{u_i \in \mathbb{R}} \left\{\frac{\partial V_i}{\partial t} + \frac{\partial V_i}{\partial q_i}(-\mu q_i + b_i u_i) + \frac{\sigma^2}{2}\frac{\partial^2 V_i}{\partial q_i^2} + (c - \bar{p})q_i + ru_i^2\right\}. \tag{7}$$

Let $V_i = k_i q_i^2 + 2s_i q_i + g_i$. Then the optimal control law is

$$\bar{u}_i = -\frac{1}{2r} b_i \frac{\partial V_i}{\partial q_i} = -\frac{b_i}{r}(k_i q_i + s_i). \tag{8}$$

From (7) we derive

$$\rho k_i = -2\mu k_i - \frac{b_i^2}{r}k_i^2, \tag{9}$$

$$\rho s_i = \frac{ds_i}{dt} + (-\mu - \frac{b_i^2}{r}k_i)s_i + \frac{c - \bar{p}}{2}, \tag{10}$$

$$\rho g_i = \frac{dg_i}{dt} - \frac{b_i^2}{r}s_i^2 + \sigma^2 k_i, \tag{11}$$

where $s_i \in C_{\rho/2}([0,\infty), \mathbb{R})$ and $g_i \in C_\rho([0,\infty), \mathbb{R})$. From (9) we solve $k_i = 0$ which is the only solution to satisfy the closed-loop stability condition $-\mu - \frac{b_i^2}{r}k_i - \frac{\rho}{2} < 0$. The growth conditions of s_i, g_i will help us determine the solution of (7).

Theorem 1. *For the optimal control problem in (2), (5) and (6), assume that $\bar{q} \in C_{\rho/2}([0,\infty), \mathbb{R})$ is given. Then we have*

(1) there exists a unique solution $s_i \in C_{\rho/2}([0,\infty), \mathbb{R})$ to (10);

(2) the optimal control law is uniquely given by $\bar{u}_i = -\frac{b_i}{r}s_i$;

(3) there exists a unique solution $g_i \in C_\rho([0,\infty), \mathbb{R})$ to (11), and the optimal cost is $V_i(0, q_i(0)) = 2s_i(0)q_0 + g_i(0)$.

Proof. Note that by (5), $\bar{q} \in C_{\rho/2}([0,\infty), \mathbb{R})$ implies $\bar{p} \in C_{\rho/2}([0,\infty), \mathbb{R})$. We can prove parts (1) and (3) by showing that $s_i(0)$ and $g_i(0)$ are uniquely determined from the fact $s_i \in C_{\rho/2}([0,\infty), \mathbb{R})$ and $g_i \in C_\rho([0,\infty), \mathbb{R})$, respectively (see e.g., Huang, 2010; Huang et al., 2007). To show part (2) we first obtain a prior integral estimate of q_i and then use the completion of squares technique (see e.g., Huang et al., 2012; Wang & Zhang, 2017). By Lemma A.1, $E\int_0^\infty e^{-\rho t}u_i^2 dt < \infty$ implies $E\int_0^\infty e^{-\rho t}q_i^2 dt < \infty$, which further ensures that $\bar{J}_i$ is well defined to be finite since $\bar{p} \in C_{\rho/2}([0,\infty), \mathbb{R})$. □

3.2. Control synthesis and analysis

Following the standard approach in mean field games (Huang et al., 2007; Wang & Zhang, 2017), we construct the equation system as follows:

$$\frac{d\bar{p}}{dt} = \alpha[\beta - \bar{p} - \bar{q}] \tag{12}$$

$$\rho s = \frac{ds}{dt} - \mu s + \frac{c - \bar{p}}{2} \tag{13}$$

$$\frac{d\bar{q}_\theta}{dt} = -\mu\bar{q}_\theta - \frac{\theta^2}{r}s \tag{14}$$

$$\bar{q} = \int_\Theta \bar{q}_\theta dF(\theta), \tag{15}$$

where $p(0) = p_0$, $q_\theta(0) = q_0$ and $s(0)$ is to be determined. Here $\bar{q}_\theta$ is regarded as the expectation of the state given the parameter θ in the individual dynamics. The last equation is due to the consistency requirement for the mean field approximation. For further analysis, we make the assumption.

(A4) There exists a solution $(s, \bar{q}_\theta, \theta \in \Theta)$ to (12)–(15) such that for each $\theta \in \Theta$, both s and $\bar{q}_\theta$ are within $C_b([0,\infty), \mathbb{R})$.

Sufficient conditions for ensuring **(A4)** may be obtained by the fixed-point methods similar to those in Huang et al. (2007) and Wang and Zhang (2017).

Proposition 1. *If $\frac{1}{2r\mu(\rho+\mu)}\int_\Theta \theta^2 dF(\theta) < 1$, then **(A4)** holds.*

Proof. By (12)–(14), we have

$$\bar{q}(t) = \int_\Theta \bar{q}_\theta(0)dF(\theta) + \int_\Theta dF(\theta)\int_0^t e^{-\mu(t-\tau_1)} \cdot\left\{-\frac{\theta^2}{r}\int_{\tau_1}^\infty e^{(\rho+\mu)(\tau_1-\tau_2)}(\mathscr{A}\bar{q})(\tau_2)d\tau_2\right\}d\tau_1 \triangleq (\mathscr{T}\bar{q})(t),$$

where $(\mathscr{A}\bar{q})(\tau_2) = \frac{1}{2}\bar{p}(0)e^{-\alpha\tau_2} + \frac{1}{2}\int_0^{\tau_2} e^{-\alpha(\tau_2-\tau_3)}(\alpha\beta - \alpha\bar{q}(\tau_3))d\tau_3 - \frac{c}{2}$. It can be verified that $\mathscr{T}$ is a contraction from the Banach space $C_b([0,\infty), \mathbb{R})$ to itself. Hence it has a unique fixed point $\bar{q} \in C_b([0,\infty), \mathbb{R})$. □

3.2.1. The case of uniform agents

We now consider the case of uniform agents, i.e., $b_i \equiv b$, $i = 1, \ldots, N$, and denote

$$z = \begin{bmatrix}\bar{p}\\ \bar{q}\\ s\end{bmatrix}, \quad M = \begin{bmatrix}-\alpha & -\alpha & 0\\ 0 & -\mu & -\frac{b^2}{r}\\ \frac{1}{2} & 0 & \rho+\mu\end{bmatrix}, \quad \bar{b} = \begin{bmatrix}\alpha\beta\\ 0\\ -\frac{c}{2}\end{bmatrix}.$$

Then

$$\frac{dz}{dt} = Mz + \bar{b}. \tag{16}$$

By direct computations, we have $\det(M) = \alpha\mu(\rho+\mu) + \frac{\alpha b^2}{2r}$ and the equation $M\bar{z} + \bar{b} = 0$ has the solution

$$\bar{z} = \left[\frac{2r\beta\mu(\rho+\mu) + b^2 c}{2r\mu(\rho+\mu) + b^2}, \frac{b^2(\beta - c)}{2r\mu(\rho+\mu) + b^2}, \frac{r\mu(c-\beta)}{2r\mu(\rho+\mu) + b^2}\right]^T.$$

Note that

$$\det(\lambda I - M) = \lambda^3 + (\alpha - \rho)\lambda^2 - (\mu^2 + \rho\mu + \alpha\rho)\lambda - \alpha\mu(\rho+\mu) - \frac{\alpha b^2}{2r}. \quad (17)$$

In what follows, we use Routh's stability criterion (Dorf & Bishop, 1998) to determine the number of roots of $|\lambda I - M| = 0$ with negative real parts. The first column of the Routh array for $|\lambda I - M|$ is $-1,\ \rho - \alpha,\ \mu^2 + \rho\mu + \alpha\rho + \frac{2r\alpha\mu(\rho+\mu)+\alpha b^2}{2r(\rho-\alpha)},\ \alpha\mu(\rho+\mu) + \frac{\alpha b^2}{2r}$. It can be verified that the first column of the Routh array always has a sign change. By Routh's stability criterion, (17) has a root with a positive real part, and two roots λ_1, λ_2 with negative real parts.

For λ_1, λ_2, let ξ_1, ξ_2 be the corresponding (generalized) complex eigenvectors. Let $\tilde{z} = z - \bar{z}$. The solution to Eq. (16) given by $\bar{z} + e^{Mt}\tilde{z}(0)$ is in $C_b([0,\infty), \mathbb{R}^3)$ if and only if there exist constants a_1, a_2 such that $\tilde{z}(0) = a_1\xi_1 + a_2\xi_2$.

Denote $\xi_1 = [(\xi_1^\dagger)^T, \xi_1^\ddagger]^T$, $\xi_2 = [(\xi_2^\dagger)^T, \xi_2^\ddagger]^T$ and $\bar{z} = [(z^\dagger)^T, z^\ddagger]^T$, where $\xi_1^\ddagger, \xi_2^\ddagger, z^\ddagger \in \mathbb{C}$. Then we have

$$a_1\xi_1^\dagger + a_2\xi_2^\dagger = [\tilde{\bar{p}}(0), \tilde{\bar{q}}(0)]^T. \quad (18)$$

Note that $[\tilde{\bar{p}}(0), \tilde{\bar{q}}(0)]^T = [p_0, q_0]^T - z^\dagger$ is given. There exists a unique solution (a_1, a_2) to (18) if and only if $\xi_1^\dagger$ and $\xi_2^\dagger$ are linearly independent.

From the analysis above, we have the following result.

Proposition 2. *(16) admits a unique solution $(s, \bar{q})$ such that s and $\bar{q}$ are in $C_b([0,\infty), \mathbb{R})$ if and only if $\xi_1^\dagger$ and $\xi_2^\dagger$ are linearly independent. In this case, **(A4)** holds.* □

Example 1. Take parameters as $[\alpha\ \beta\ \mu\ b_i\ \sigma\ \rho\ r] = [1\ 10\ 0.15\ 1\ 0.2\ 0.6\ 1]$. Take $p(0) = 1$ and $q_i(0)$ having normal distribution $N(2, 0.2)$. In this case, M has only two eigenvalues with negative real parts $-0.6875 + 0.3944i$ and $-0.6875 - 0.3944i$. The corresponding eigenvectors are $[-0.8557,\ 0.2674+0.3375i,\ 0.2768 + 0.0759i]^T$ and $[-0.8557,\ 0.2674 - 0.3375i,\ 0.2768 - 0.0759i]^T$, respectively. By (18), we have $a_1 = 1.4429 + 7.8552i$ and $a_2 = 1.4429 - 7.8552i$. Then (16) admits a unique solution in $C_b([0,\infty), \mathbb{R}^3)$. However, $\frac{b^2}{2r\mu(\rho+\mu)} = 4.444 > 1$. This example satisfies the condition of Proposition 2, but not that of Proposition 1.

3.3. ε-Nash equilibrium

Consider the system of N firms. Let the control strategy of firm i be given by

$$\hat{u}_i = -\frac{b_i}{r}s, \quad i = 1, \ldots, N, \quad (19)$$

where $s \in C_b([0,\infty), \mathbb{R})$ is determined by (12)–(15). After the strategy (19) is applied, the closed-loop dynamics for firm i may be written as follows:

$$\frac{d\hat{p}(t)}{dt} = -\alpha\hat{p}(t) - \alpha\hat{q}^{(N)}(t) + \alpha\beta, \quad (20)$$

$$d\hat{q}_i(t) = -\mu\hat{q}_i(t)dt - \frac{b_i^2}{r}s(t)dt + \sigma dw_i,\ i = 1, \ldots, N. \quad (21)$$

Denote $\varepsilon_N = \left|\int_\Theta \theta^2 dF_N(\theta) - \int_\Theta \theta^2 dF(\theta)\right|$.

Lemma 1. *For the system (1)–(3), if assumptions **(A1)–(A4)** hold, then the closed-loop system (20)–(21) satisfies*

$$\sup_{t\ge 0, N\ge 1} E\left\{|\hat{p}(t)|^2 + |\hat{q}^{(N)}(t)|^2\right\} \le C_0, \quad (22)$$

$$\sup_{t\ge 0} E\left\{|\hat{p} - \bar{p}|^2 + |\hat{q}^{(N)} - \bar{q}|^2\right\} = O(\varepsilon_N^2 + 1/N). \quad (23)$$

Proof. By (21), it follows that

$$d\hat{q}^{(N)}(t) = \left[-\mu\hat{q}^{(N)}(t) - \frac{1}{N}\sum_{i=1}^N \frac{b_i^2}{r}s(t)\right]dt + \frac{1}{N}\sum_{i=1}^N \sigma dw_i(t).$$

Note that $[\hat{p}, \hat{q}^{(N)}]^T$ and $[\hat{p} - \bar{p}, \hat{q}^{(N)} - \bar{q}]^T$ satisfy two linear SDEs with the same state coefficient matrix $G = \begin{bmatrix} -\alpha & -\alpha \\ 0 & -\mu \end{bmatrix}$. We obtain (22) and (23) by elementary SDE estimates. □

We proceed to show an asymptotic Nash equilibrium property. Denote $\hat{u}_{-i} = (\hat{u}_1, \ldots, \hat{u}_{i-1}, \hat{u}_{i+1}, \ldots, \hat{u}_N)$, and

$$\mathscr{U}_c = \left\{u_i : u_i(t) \text{ is adapted to } \sigma\{\cup_{j=1}^N \mathscr{F}_t^j\},\ E\int_0^\infty e^{-\rho t}u_i^2(t)dt < \infty\right\}.$$

Theorem 2. *For the problem (1)–(2), assume that **(A1)–(A4)** hold. Then the set of strategies $(\hat{u}_1, \ldots, \hat{u}_N)$ given by (19) is an ε-Nash equilibrium, i.e.,*

$$J_i(\hat{u}_i, \hat{u}_{-i}) - \varepsilon \le \inf_{u_i \in \mathscr{U}_c} J_i(u_i, \hat{u}_{-i}) \le J_i(\hat{u}_i, \hat{u}_{-i}), \quad (24)$$

where $\varepsilon = O(\varepsilon_N + \frac{1}{\sqrt{N}})$.

Proof. See Appendix A. □

4. Social solutions to output adjustment

We first construct an auxiliary optimal control problem by examining the social cost variation due to the control perturbation of a single agent. Then, by mean field approximations we design a set of decentralized strategies.

4.1. An auxiliary optimal control problem

Lemma 2. *$J_{\rm soc}^{(N)}(u)$ is coercive with respect to $(u_1, \ldots, u_N)$, i.e., there exist constants $C_2 > 0$ and $C_3 > 0$ such that*

$$J_{\rm soc}^{(N)}(u) \ge \frac{C_2}{N}E\int_0^\infty e^{-\rho t}\sum_{i=1}^N u_i^2 dt - C_3.$$

Proof. From Lemma B.2, we get the lemma immediately. □

This coercivity property ensures the existence of a centralized optimal solution $(\widehat{u}_1, \ldots, \widehat{u}_N)$ to the social optimum problem in (1)–(2) and (4). Let $\widehat{q}_i$ be generated by $\widehat{u}_i$. We now derive an auxiliary optimal control problem from the original social optimum problem by perturbing the strategy of a fixed agent. Denote the control problem **(P1)**:

$$\begin{aligned}
\frac{dp}{dt} &= -\alpha p - \frac{\alpha}{N}q_i - \alpha\widehat{q}_{-i}^{(N)} + \alpha\beta, \\
dq_i &= -\mu q_i dt + b_i u_i dt + \sigma dw_i, \\
\frac{dv_i}{dt} &= -\alpha v_i - \alpha q_i,\ v_i(0) = 0, \\
J_i^*(u_i) &= E\int_0^\infty e^{-\rho t}\left[(c - p)q_i - v_i\widehat{q}_{-i}^{(N)} + ru_i^2\right]dt,
\end{aligned}$$

where $\widehat{q}_{-i}^{(N)} = \frac{1}{N}\sum_{j=1, j\ne i}^N \widehat{q}_j$ and $\widehat{u}_{-i}$ is fixed.

Lemma 3. *If $\widehat{u} = (\widehat{u}_1, \ldots, \widehat{u}_N)$ minimizes $J_{\rm soc}^{(N)}$ where each $\widehat{u}_i \in \mathscr{U}_c$, then $\widehat{u}_i$ is necessarily the optimal control of **(P1)**.*

Proof. It follows from (3) that

$$J_{\rm soc}^{(N)} = \frac{1}{N}\sum_{i=1}^{N} J_i(\widehat{u}_1,\ldots,\widehat{u}_{i-1},u_i,\widehat{u}_{i+1},\ldots,\widehat{u}_N) = \frac{1}{N}E\int_0^\infty e^{-\rho t}(Y_i+Y_i')dt,$$

where

$$Y_i=(c-p)q_i-p\sum_{j=1,j\neq i}^{N}\widehat{q}_j+ru_i^2,$$

$$Y_i'=c\sum_{j=1,j\neq i}^{N}\widehat{q}_j+\sum_{j=1,j\neq i}^{N}r\widehat{u}_j^2.$$

From (1) and the argument in Huang et al. (2012, Lemma 3.5), $Y_i=Z_i+Z_i'$ where

$$Z_i=(c-p)q_i+\alpha\int_0^t e^{-\alpha(t-\tau)}q_id\tau\cdot\widehat{q}_{-i}^{(N)}+ru_i^2,$$

$$Z_i'=-\Big[e^{-\alpha t}p(0)+\int_0^t e^{-\alpha(t-\tau)}(\alpha\beta-\alpha\widehat{q}_{-i}^{(N)})d\tau\Big]\sum_{j=1,j\neq i}^{N}\widehat{q}_j.$$

Note that $J_{\rm soc}^{(N)}=\frac{1}{N}E\int_0^\infty e^{-\rho t}(Z_i+Z_i'+Y_i')dt$, where neither Z_i' nor Y_i' changes with u_i. Thus, for agent i minimizing $J_{\rm soc}^{(N)}$ is equivalent to minimizing $E\int_0^\infty e^{-\rho t}Z_idt$, which in turn is equal to $J_i^*(u_i)$. □

4.2. Mean field approximation

To approximate Problem (P1) for large N, we construct the auxiliary limiting optimal control problem **(P2)**:

$$\begin{bmatrix} d\bar{p}\\ dq_i\\ dv_i\end{bmatrix}=\begin{bmatrix}-\alpha&0&0\\0&-\mu&0\\0&-\alpha&-\alpha\end{bmatrix}\begin{bmatrix}\bar{p}\\ q_i\\ v_i\end{bmatrix}dt+\begin{bmatrix}0\\ b_i\\ 0\end{bmatrix}u_idt + \begin{bmatrix}\alpha\beta-\alpha\bar{q}\\0\\0\end{bmatrix}dt+\begin{bmatrix}0\\1\\0\end{bmatrix}dw_i,\quad \begin{bmatrix}\bar{p}(0)\\ q_i(0)\\ v_i(0)\end{bmatrix}=\begin{bmatrix}p_0\\ q_i(0)\\ 0\end{bmatrix} \tag{25}$$

with cost function

$$\bar{J}_i^*(u_i)=E\int_0^\infty e^{-\rho t}[(c-\bar{p})q_i-\bar{q}v_i+ru_i^2]dt. \tag{26}$$

Here $\bar{q}\in C_{\rho/2}([0,\infty),\mathbb{R})$ is an approximation of $q^{(N)}$.

For the system (25)–(26), we take $\bar{p}$ as an exogenous signal.

Lemma 4. *$\bar{J}_i^*(u_i)$ is strictly convex and coercive.*

Proof. For the system (25)–(26), the state is (q_i,v_i); $\bar{p}$ and $\bar{q}$ do not depend on u_i. We can directly show that $\bar{J}_i^*(u_i)$ is strictly convex in u_i. Following the proof of Proposition A.1, we can show that $\bar{J}_i^*(u_i)$ is coercive. □

Let

$$A=\begin{bmatrix}-\mu&0\\-\alpha&-\alpha\end{bmatrix},\quad B_i=\begin{bmatrix}b_i\\0\end{bmatrix}.$$

By Theorem 1, Lemma 4 and Kurdila and Zabarankin (2006), Problem (P2) has the unique optimal control given by $u_i=-\frac{1}{r}B_i^T\check{s}$, where $\check{s}\in C_{\rho/2}([0,\infty),\mathbb{R}^2)$ is determined from

$$\rho\check{s}=\frac{d\check{s}}{dt}+A^T\check{s}+\frac{1}{2}[c-\bar{p},-\bar{q}]^T.$$

Let $B_\theta=[\theta,0]^T$. When $\theta=b_i$, we use B_θ to indicate B_i by a slight abuse of notation. Following the standard approach in Huang et al. (2007) and Huang et al. (2012), we construct the equation system:

$$\frac{d\bar{p}}{dt}=\alpha[-\bar{p}+\beta-\bar{q}], \tag{27}$$

$$\rho\check{s}=\frac{d\check{s}}{dt}+A^T\check{s}+\frac{1}{2}[c-\bar{p},-\bar{q}]^T, \tag{28}$$

$$\frac{dy_\theta}{dt}=Ay_\theta-\frac{1}{r}B_\theta B_\theta^T\check{s}, \tag{29}$$

$$\bar{q}=[1,0]\int_\Theta y_\theta dF(\theta), \tag{30}$$

where $y_\theta(0)=[q_0,0]^T$. Here y_θ is regarded as the expectation of $[q_\theta,v_\theta]^T$ given the parameter $b_i=\theta$ in the individual dynamics. For further analysis, we make the following assumption.

(A5) There exists a solution $(\check{s},y_\theta,\theta\in\Theta)$ to (27)–(30) such that for any $\theta\in\Theta$, both $\check{s}$ and y_θ are within $C_b([0,\infty),\mathbb{R}^2)$.

For the case of uniform agents ($b_i\equiv b$), the equation system (27)–(30) reduces to

$$\frac{d\bar{p}}{dt}=\alpha[\beta-\bar{p}-\bar{q}], \tag{31}$$

$$\rho\check{s}=\frac{d\check{s}}{dt}+A^T\check{s}+\frac{1}{2}[c-\bar{p},-\bar{q}]^T, \tag{32}$$

$$\frac{dy}{dt}=Ay-\frac{1}{r}BB^T\check{s}, \tag{33}$$

where $\bar{p}(0)=p_0$, $y(0)=[q_0,0]^T$ and $B=[b,0]^T$. Denote $\varphi=[\bar{p},\check{s}^T,y^T]^T$ and $\bar{b}_s=[\alpha\beta,-\frac{c}{2},0,0,0]^T$. Then we have

$$\frac{d\varphi}{dt}=M_s\varphi+\bar{b}_s, \tag{34}$$

where M_s is a 5×5 matrix. By straightforward computation, we can show $M_s\varphi+\bar{b}_s=0$ has a unique solution, denoted as z_s. Furthermore, we obtain

$$\det(\lambda I-M_s)=(\lambda+\alpha)\times\Big[\lambda^4-2\rho\lambda^3 + (\rho^2-(\alpha+\mu)\rho-\alpha^2-\mu^2)\lambda^2+\rho[(\alpha+\mu)\rho+\alpha^2+\mu^2]\lambda + \alpha\mu(\rho+\alpha)(\rho+\mu)+\frac{\alpha b^2}{2r}(\rho+2\alpha)\Big]. \tag{35}$$

By Routh's stability criterion (Dorf & Bishop, 1998), (35) has two roots with positive real parts, and three roots $\check{\lambda}_1,\check{\lambda}_2,\check{\lambda}_3$ with negative real parts. For $\check{\lambda}_1,\check{\lambda}_2,\check{\lambda}_3$, let ζ_1,ζ_2,ζ_3 be the corresponding (generalized) complex eigenvectors. Let $\tilde{\varphi}(0)=[\tilde{\bar{p}}(0),\tilde{\check{s}}^T(0),\tilde{y}^T(0)]^T=\varphi(0)-z_s$. The solution to Eq. (34) given by $z_s+e^{M_st}\tilde{\varphi}(0)$ is in $C_b([0,\infty),\mathbb{R}^5)$ if and only if there exist constants $\check{a}_1,\check{a}_2,\check{a}_3$ such that $\tilde{\varphi}(0)=\check{a}_1\zeta_1+\check{a}_2\zeta_2+\check{a}_3\zeta_3$.

Denote $\zeta_i=[\zeta_i^\dagger,(\zeta_i^\ddagger)^T,(\zeta_i^\S)^T]^T$ and $\vartheta_i=[(\zeta_i^\dagger)^T,(\zeta_i^\S)^T]^T$, where $\zeta_i^\dagger\in\mathbb{C}$ and $\zeta_i^\ddagger,\zeta_i^\S\in\mathbb{C}^2$, $i=1,2,3$. Then we have

$$\check{a}_1\vartheta_1+\check{a}_2\vartheta_2+\check{a}_3\vartheta_3=[\tilde{\bar{p}}(0),\tilde{y}(0)]^T. \tag{36}$$

Note that $[\tilde{\bar{p}}(0),\tilde{y}(0)]^T$ is given. Then (36) admits a unique solution if and only if ϑ_1, ϑ_2 and ϑ_3 are linearly independent.

From the analysis above, we have the following result.

Proposition 3. *(A5) holds if (36) admits a solution. In particular, (31)–(33) admits a unique solution $(\check{s},y_\theta)$ such that $\check{s}$ and y_θ are within $C_b([0,\infty),\mathbb{R}^2)$ if and only if ϑ_1, ϑ_2 and ϑ_3 are linearly independent.* □

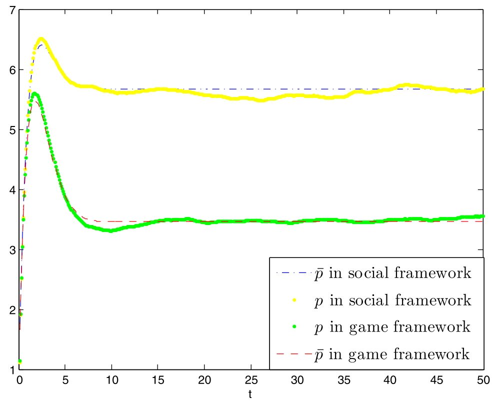

Fig. 1. Curves of p and $\bar{p}$ in the game and social optimum.

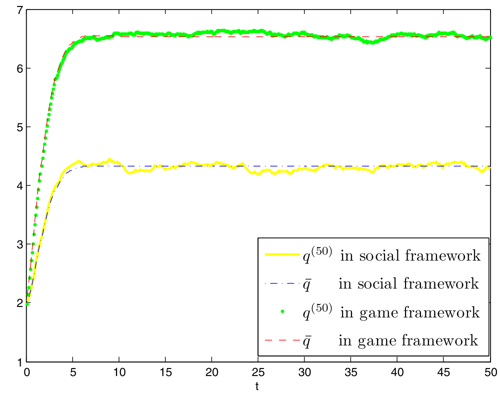

Fig. 2. Curves of $q^{(50)}$, $\bar{q}$ in the game and social optimum.

4.3. Asymptotic optimality

Consider the system of N firms. Let the control strategy of firm i be given by

$$\check{u}_i = -\frac{1}{r}B_i^T\check{s}, \quad i = 1, \ldots, N, \tag{37}$$

where $\check{s} \in C_b([0,\infty), \mathbb{R}^2)$ is determined by the equation system (27)–(30). After the strategy (37) is applied, the closed-loop dynamics for firm i may be written as follows:

$$\frac{d\check{p}}{dt} = -\alpha\check{p} - \alpha\check{q}^{(N)} + \alpha\beta, \tag{38}$$

$$d\check{q}_i = -\mu\check{q}_i - \left[\frac{b_i^2}{r}, 0\right]\check{s}dt + \sigma dw_i, \tag{39}$$

$$\frac{d\check{v}_i}{dt} = -\alpha\check{q}_i - \alpha\check{v}_i, \tag{40}$$

where $\check{p}(0) = p_0$, $\check{q}_i(0) = q_i(0)$, and $\check{v}_i(0) = 0$.

Theorem 3. *Assume that **(A1)–(A3)** and **(A5)** hold. The set of strategies* $\{\check{u}_i = -\frac{1}{r}B_i^T\check{s}, i = 1, \ldots, N\}$ *has asymptotic social optimality, i.e.,*

$$\left| J_{\rm soc}^{(N)}(\check{u}) - \inf_{u_i \in \mathcal{U}_c} J_{\rm soc}^{(N)}(u) \right| = O\left(\frac{1}{\sqrt{N}} + \varepsilon_N\right).$$

Proof. See Appendix B. □

5. Numerical simulation

We give a numerical example to compare two solutions.

Take $[\alpha,\ \beta,\ \mu,\ b_i,\ \sigma,\ \rho\ r] = [1,\ 10,\ 0.15,\ 1,\ 0.2,\ 0.6\ 1]$, $p(0) = 1$, and $q_i(0)$ with normal distribution $N(2, 0.2)$. By (18), we have $a_1 = 1.4429 + 7.8552i$ and $a_2 = 1.4429 - 7.8552i$; by (36), we get $\check{a}_1 = 9$, $\check{a}_2 = -3.5906 - 6.5046i$ and $\check{a}_3 = -3.5906 + 6.5046i$. This determines s and $\check{s}$.

Fig. 1 depicts the curves of p and $\bar{p}$ within game and social frameworks for a total of 50 agents. Fig. 2 shows the curves of $q^{(50)}$ and $\bar{q}$. The curves of p and $\bar{p}$ (respectively, of $q^{(50)}$ and $\bar{q}$) are very close to each other, which indicates tight mean field approximations. From the game theoretic solution to the social optimum solution, the price gets a significant increase, and the average output becomes lower. This is similar to a well known phenomenon that the collusion of firms results in a higher price and smaller outputs (Varian, 1993 p. 467).

By Theorem 1, we have the asymptotic Nash cost $J_{\rm Nash}^{(\infty)} \triangleq \lim_{N\to\infty} J_i(\hat{u}_i, \hat{u}_{-i}) = 2s(0)q_0 + g(0)$, where $g(0) = -\int_0^\infty e^{-\rho t}s^2 dt$.

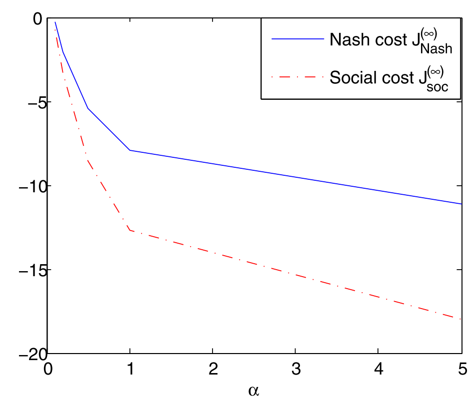

Fig. 3. Curves of $J_{\rm Nash}^{(\infty)}$ and $J_{\rm soc}^{(\infty)}$ with respect to α.

Table 1
Optimal costs under game and social solutions.

α	0.1	0.2	0.5	1	5
$J_{\rm Nash}^{(\infty)}$	−0.31	−2.09	−5.47	−7.95	−11.11
$J_{\rm soc}^{(\infty)}$	−0.78	−3.32	−8.59	−12.72	−18.02

Note that $s = z^\ddagger + a_1 e^{\lambda_1 t}\xi_1^\ddagger + a_2 e^{\lambda_2 t}\xi_2^\ddagger$. Then $J_{\rm Nash}^{(\infty)}$ can be explicitly calculated. By using a derivation similar to Theorem 6.2 in Huang et al. (2012), we obtain the asymptotic social optimum cost $J_{\rm soc}^{(\infty)} \triangleq \lim_{N\to\infty} J_{\rm soc}^{(N)}(\check{u}) = 2\check{s}_1(0)q_0 + \check{g}(0) + \int_0^\infty e^{-\rho t}\bar{q}\bar{v}dt$, where $\check{s}_1 = [1, 0]\check{s}$ and $\check{g}(0) = -\int_0^\infty e^{-\rho t}\check{s}_1^2 dt$. From (35), we obtain $\check{\lambda}_1 = -\alpha$ and $\zeta_1 = [0, 0, 0, 0, 1]^T$. Thus, we can find the explicit form of $J_{\rm soc}^{(\infty)}$.

The comparison of the two costs is shown in Table 1 and Fig. 3. It can be seen that $J_{\rm Nash}^{(\infty)}$ is greater than $J_{\rm soc}^{(\infty)}$, and more so when α increases. This illustrates that the cooperation of firms leads to a drop in costs.

6. Concluding remarks

This paper studies mean field production output optimization with sticky prices and adjustment costs, and provides a comparison of the noncooperative and cooperative solutions. For future work, it is of interest to consider dynamic production output competition with noisy sticky prices.

Appendix A. Proof of Theorem 2

Proposition A.1. *Under **(A1)–(A3)**, $\bar{J}_i(u_i)$ is coercive in u_i, i.e., for some $\varepsilon_0 > 0$,*

$$\bar{J}_i(u_i) \geq \varepsilon_0 \|u_i\|_\rho^2 - C, \tag{A.1}$$

where C depends on p_0, q_0 and $\varepsilon_0 > 0$.

Proof. From (2),

$$q_i(t) = q_i(0)e^{-\mu t} + \int_0^t e^{-\mu(t-\tau)}\big[b_i u_i(\tau)d\tau + \sigma dw_i(\tau)\big].$$

Thus, by Cauchy's inequality and **(A3)**,

$$\begin{aligned} &E\int_0^\infty e^{-\rho t}|q_i(t)|dt \\ \leq& C + E\int_0^\infty e^{-(\rho+\mu)t}\int_0^t e^{\mu\tau}|b_i u_i(\tau)|d\tau dt \\ =& C + E\int_0^\infty e^{\mu\tau}|b_i u_i(\tau)|\int_\tau^\infty e^{-(\rho+\mu)t}dtd\tau \\ \leq& C + \delta_1 E\int_0^\infty e^{-\rho\tau}|u_i(\tau)|^2 d\tau, \end{aligned} \tag{A.2}$$

where δ_1 is a sufficiently small positive number. Note that $\bar{q} \in C_b([0,\infty),\mathbb{R})$. By (5), it follows that $\bar{p} \in C_b([0,\infty),\mathbb{R})$. From this together with (A.2) and (6), we obtain (A.1). □

Lemma A.1. *For any $i = 1,\ldots,N$, there exists a constant C such that*

$$\|q_i\|_\rho^2 \leq C\|u_i\|_\rho^2 + C. \tag{A.3}$$

Proof. Denote $q_{i,\rho} = e^{-\frac{\rho}{2}}q_i$ and $u_{i,\rho} = e^{-\frac{\rho}{2}}u_i$. Then

$$dq_{i,\rho} = -(\mu + \frac{\rho}{2})q_{i,\rho}dt + u_{i,\rho}dt + e^{-\frac{\rho}{2}}dw_i.$$

As in the proof of Lemma A.1 in Huang (2010), we get (A.3). □

Lemma A.2. *For (1)–(2), assume that **(A1)–(A4)** hold. For $u_i \in \mathscr{U}_c$, if $J(u_i,\hat{u}_{-i}) \leq C_1$, then there exist an integer N_0 and a constant C_2 such that for all $N \geq N_0$, $\|u_i\|_\rho^2 \leq C_2$.*

Proof. For $u_i \in \mathscr{U}_c$, we have

$$\begin{aligned} J_i(u_i,\hat{u}_{-i}) &= \bar{J}_i(u_i) - E\int_0^\infty e^{-\rho t}[p(t)-\bar{p}(t)]q_i(t)dt \\ &\geq \bar{J}_i(u_i) - \Big[E\int_0^\infty e^{-\rho t}|p(t)-\bar{p}(t)|^2dt \\ &\quad \cdot E\int_0^\infty e^{-\rho t}|q_i(t)|^2dt\Big]^{1/2} \\ &\triangleq \bar{J}_i(u_i) - I_1. \end{aligned} \tag{A.4}$$

By Proposition A.1, $\bar{J}_i(u_i)$ is coercive, i.e.,

$$\bar{J}_i(u_i) \geq \varepsilon_0\|u_i\|_\rho^2 - C. \tag{A.5}$$

We now estimate I_1. As in the proof of Lemma A.1, we use elementary estimates to the SDE of $[p-\hat{p}, q^{(N)}-\hat{q}^{(N)}]^T$ to obtain

$$\int_0^\infty e^{-\rho t}|p(t)-\hat{p}(t)|^2dt \leq \frac{C}{N^2} + \frac{C}{N^2}\|u_i\|_\rho^2. \tag{A.6}$$

From this together with Lemma 1, it follows that

$$E\int_0^\infty e^{-\rho t}|p(t)-\bar{p}(t)|^2dt \leq C + \frac{C}{N^2}\|u_i\|_\rho^2,$$

which implies $|I_1| \leq C + \frac{C}{N}\|u_i\|_\rho^2$ in view of Lemma A.1. Combining this together with (A.4) and (A.5) yields

$$C_1 \geq J_i(u_i,\hat{u}_{-i}) \geq (\varepsilon_0 - \frac{C}{N})\|u_i\|_\rho^2 - 2C.$$

Let $N_0 = \inf\{m \in \mathbb{Z} | m > C/\varepsilon_0\}$. So there exists C_2 such that for all $N \geq N_0$, $\|u_i\|_\rho^2 \leq \frac{N(C_1+2C)}{N\varepsilon_0 - C} \leq C_2$. □

Proof of Theorem 2. It suffices to show the first inequality in (24) under $J_i(u_i,\hat{u}_{-i}) \leq C_1$. By Lemma A.2, $J_i(u_i,\hat{u}_{-i}) \leq C_1$ implies $\|u_i\|_\rho^2 \leq C_2$. It follows from (A.4) that

$$J_i(u_i,\hat{u}_{-i}) \geq \bar{J}_i(u_i) - I_1, \tag{A.7}$$

where I_1 is given by (A.4). By (A.6) and Lemma 1,

$$E\int_0^\infty e^{-\rho t}|p(t)-\bar{p}(t)|^2dt \leq O(\varepsilon_N^2 + \frac{1}{N}),$$

which further implies $|I_1| \leq O(\varepsilon_N + \frac{1}{\sqrt{N}})$ due to Lemma A.1. Thus, by (A.7) it follows that

$$J_i(u_i,\hat{u}_{-i}) \geq \bar{J}_i(\hat{u}_i) - O(\varepsilon_N + \frac{1}{\sqrt{N}}). \tag{A.8}$$

On the other hand, by Schwarz's inequality and Lemma 1,

$$\begin{aligned} &|\bar{J}_i(\hat{u}_i) - J_i(\hat{u}_i,\hat{u}_{-i})| = E\int_0^\infty e^{-\rho t}|\hat{p}(t)\hat{q}_i(t) - \bar{p}(t)\hat{q}_i(t)|dt \\ \leq& O(\varepsilon_N + \frac{1}{\sqrt{N}}). \end{aligned}$$

From this together with (A.8), (24) follows. □

Appendix B. Proof of Theorem 3

Lemma B.1. *Assume that **(A1)–(A3)** and **(A5)** hold. For the set of strategies $\{\check{u}_i, i=1,\ldots,N\}$, we have*

$$\sup_{t\geq 0, N\geq 1} E\Big\{|\check{p}|^2 + |\check{q}_i|^2 + |\check{q}^{(N)}|^2 + |\check{v}_i|^2\Big\} \leq C_0, \tag{B.1}$$

$$\sup_{t\geq 0} E\Big\{|\check{p}-\bar{p}|^2 + |\check{q}^{(N)}-\bar{q}|^2 + |\check{v}^{(N)}-\bar{v}|^2\Big\} = O(\varepsilon_N^2 + 1/N), \tag{B.2}$$

where $\bar{v} = [0,1]\int_\Theta y_\theta dF(\theta)$, and $\check{v}^{(N)} = \frac{1}{N}\sum_{i=1}^N \check{v}_i$.

Proof. It follows from (39) and (40) that

$$\begin{bmatrix} d\check{q}_i \\ d\check{v}_i \end{bmatrix} = A\begin{bmatrix} \check{q}_i \\ \check{v}_i \end{bmatrix}dt - \frac{1}{r}B_iB_i^T\check{s}dt + \begin{bmatrix} dw_i \\ 0 \end{bmatrix}.$$

Note that A is Hurwitz. By **(A3)**, there exists a constant C_0 independent of (i,N) such that $\sup_{t\geq 0}E\big\{|\check{q}_i(t)|^2 + |\check{v}_i(t)|^2\big\} \leq C_0$, which further gives $\sup_{t\geq 0}E\big\{|\check{q}^{(N)}(t)|^2\big\} \leq C_0$. This with (38) leads to (B.1). Denote $\xi = [\check{q}^{(N)}-\bar{q}, \check{v}^{(N)}-\bar{v}]^T$. By using basic estimates for the SDE of ξ, (B.2) follows. □

Lemma B.2. *There exist constants $C_1 > 0$, $C_2 > 0$ and $C_3 > 0$ such that*

$$J_{soc}^{(N)}(u) \geq C_1\int_0^\infty e^{-\rho t}p^2dt + \frac{C_2}{N}\int_0^\infty e^{-\rho t}\sum_{i=1}^N u_i^2dt - C_3.$$

Proof. By **(A3)**, assume $|b_i| \leq \hat{b}$, $i=1,\ldots,N$. By (2),

$$\begin{aligned} \Big|q^{(N)}(t)\Big| \leq& \Big|q^{(N)}(0)e^{-\mu t}\Big| + \int_0^t e^{-\mu(t-\tau)}\Big|\frac{1}{N}\sum_{i=1}^N b_iu_i(\tau)\Big|d\tau \\ &+ \Big|\frac{\sigma}{N}\sum_{i=1}^N\int_0^t e^{-\mu(t-\tau)}dw_i(\tau)\Big|. \end{aligned}$$

Thus, by Cauchy's inequality,

$$\begin{aligned}
&E\int_0^\infty e^{-\rho t}|q^{(N)}|dt\\
\le& C+E\int_0^\infty e^{-(\rho+\mu)t}\int_0^t e^{\mu\tau}\Big|\frac{1}{N}\sum_{i=1}^N b_iu_i(\tau)\Big|d\tau dt\\
=& C+E\int_0^\infty e^{\mu\tau}\Big|\frac{1}{N}\sum_{i=1}^N b_iu_i(\tau)\Big|\int_\tau^\infty e^{-(\rho+\mu)t}dtd\tau\\
\le& C+\frac{\delta_1\hat{b}^2}{N}E\int_0^\infty e^{-\rho\tau}\sum_{i=1}^N u_i^2(\tau)d\tau,
\end{aligned}\tag{B.3}$$

where $\delta_1>0$ is sufficiently small. By (1) and (B.3),

$$\begin{aligned}
&E\int_0^\infty e^{-\rho t}|p(t)|dt\\
\le& C+\alpha E\int_0^\infty e^{-(\rho+\alpha)t}\int_0^t e^{\alpha\tau}|q^{(N)}(\tau)|d\tau dt\\
=& C+\frac{\alpha}{\rho+\alpha}E\int_0^\infty e^{-\rho\tau}|q^{(N)}(\tau)|d\tau\\
\le& C+\frac{\alpha\delta_1\hat{b}^2}{N(\rho+\alpha)}E\int_0^\infty e^{-\rho\tau}\sum_{i=1}^N u_i^2(\tau)d\tau.
\end{aligned}\tag{B.4}$$

Consider the system

$$\begin{aligned}
\frac{dp}{dt}&=-\alpha p-\alpha\acute{u},\ p(0)=p_0\\
y&=-p,\quad \acute{u}=q^{(N)}-\beta.
\end{aligned}\tag{B.5}$$

By verifying the conditions in the positive real lemma (see e.g., Khalil, 1996, Lemma 6.2), we claim that the transfer function of the system (B.5) is positive real, which leads to the passivity of (B.5). This implies that there exists a constant $l>0$ such that $\acute{u}y\ge\dot{V}(p)$ where $V(p)\triangleq lp^2$, which further gives

$$\begin{aligned}
&E\int_0^\infty e^{-\rho t}(-p)(q^{(N)}-\beta)dt\ge E\int_0^\infty e^{-\rho t}d(lp^2)\\
=& lE\Big[\rho\int_0^\infty p^2e^{-\rho t}dt-p^2(0)\Big].
\end{aligned}\tag{B.6}$$

From this together with (B.3) and (B.4), we have

$$\begin{aligned}
&J_{\mathrm{soc}}^{(N)}(u)\\
=& E\int_0^\infty e^{-\rho t}\Big[(-p)(q^{(N)}-\beta)-\beta p+cq^{(N)}+\frac{r}{N}\sum_{i=1}^N u_i^2(t)\Big]dt\\
\ge& lE\Big[\rho\int_0^\infty p^2(t)e^{-\rho t}dt-p^2(0)\Big]\\
&+\frac{1}{N}\Big(r-\frac{\beta\alpha\delta_1\hat{b}^2}{\rho+\alpha}-c\delta_1\hat{b}^2\Big)E\int_0^\infty e^{-\rho t}\sum_{i=1}^N u_i^2(t)dt-C.
\end{aligned}$$

This completes the proof of the lemma. □

Proof of Theorem 3. Notice $\inf_{u_i\in\mathscr{U}_c}J_{\mathrm{soc}}^{(N)}(u)\le J^{(N)}(\check{u})\le C$. It suffices to consider all $u_i\in\mathscr{U}_c$ satisfying $J_{\mathrm{soc}}^{(N)}(u)\le J^{(N)}(\check{u})\le C$. By Lemma B.2,

$$E\int_0^\infty e^{-\rho t}\Big(|p|^2+\frac{1}{N}\sum_{i=1}^N|u_i|^2\Big)dt<\infty.\tag{B.7}$$

Let $\tilde{q}_i=q_i-\check{q}_i$, $\tilde{p}=p-\check{p}$, $\tilde{v}_i=v_i-\check{v}_i$ and $\tilde{u}_i=u_i-\check{u}_i=u_i+\frac{1}{r}[b_i,0]\check{s}$. By (1), (2) and (38)–(40) we have

$$\frac{d\tilde{q}_i}{dt}=-\mu\tilde{q}_i+b_i\tilde{u}_i,\tag{B.8}$$

$$\frac{d\tilde{p}}{dt}=-\alpha\tilde{p}-\alpha\tilde{q}^{(N)},\tag{B.9}$$

$$\frac{d\tilde{v}_i}{dt}=-\alpha\tilde{v}_i-\alpha\tilde{q}_i,\tag{B.10}$$

where $\tilde{q}^{(N)}=\frac{1}{N}\sum_{i=1}^N\tilde{q}_i$, and $\tilde{q}_i(0)=\tilde{p}(0)=\tilde{v}_i(0)=0$. Denote $\tilde{v}^{(N)}=\frac{1}{N}\sum_{i=1}^N\tilde{v}_i$. It follows from (B.10) that

$$\frac{d\tilde{v}^{(N)}}{dt}=-\alpha\tilde{v}^{(N)}-\alpha\tilde{q}^{(N)},\quad \tilde{v}^{(N)}(0)=0.$$

This together with (B.9) gives $\tilde{v}^{(N)}=\tilde{p}$. We have

$$\begin{aligned}
&J_{\mathrm{soc}}^{(N)}(u)\\
=& J_{\mathrm{soc}}^{(N)}(\check{u})+E\int_0^\infty e^{-\rho t}\Big[\frac{1}{N}\sum_{i=1}^N r\tilde{u}_i^2-\tilde{p}\tilde{q}^{(N)}\Big]dt\\
&+E\int_0^\infty e^{-\rho t}\Big[(c-\check{p})\tilde{q}^{(N)}-\tilde{p}\check{q}^{(N)}-\frac{1}{N}\sum_{i=1}^N 2\tilde{u}_iB_i^Ts\Big]dt\\
\triangleq& J_{\mathrm{soc}}^{(N)}(\check{u})+\tilde{J}_{\mathrm{soc}}^{(N)}(\tilde{u})+I^{(N)}.
\end{aligned}\tag{B.11}$$

To complete the proof of the theorem, we continue to show $\tilde{J}_{\mathrm{soc}}^{(N)}(\tilde{u})\ge0$ and $|I^{(N)}|=O(\frac{1}{\sqrt{N}}+\varepsilon_N)$. For the system

$$\begin{aligned}
\frac{d\tilde{p}}{dt}&=-\alpha\tilde{p}-\alpha u,\quad \tilde{p}(0)=0\\
y&=-\tilde{p},\quad u=\tilde{q}^{(N)},
\end{aligned}$$

by the positive real lemma (see e.g., Khalil, 1996), it is passive. Thus, there exists a constant $\tilde{l}>0$ such that

$$E\int_0^\infty e^{-\rho t}(-\tilde{p})\tilde{q}^{(N)}dt\ge\tilde{l}\rho E\Big[\int_0^\infty e^{-\rho t}\tilde{p}^2dt\Big]\ge0,\tag{B.12}$$

which combined with (B.11) leads to $\tilde{J}_{\mathrm{soc}}^{(N)}(\tilde{u})\ge0$. We have

$$\begin{aligned}
I^{(N)}=&\int_0^\infty e^{-\rho t}\Big[(c-\bar{p})\tilde{q}^{(N)}-\tilde{p}\bar{q}-\frac{1}{N}\sum_{i=1}^N 2\tilde{u}_iB_i^Ts\Big]dt\\
&+\int_0^\infty e^{-\rho t}\big[(\bar{p}-\check{p})\tilde{q}^{(N)}+\tilde{p}(\bar{q}-\check{q}^{(N)})\big]dt\\
\triangleq&\zeta_1+\zeta_2.
\end{aligned}\tag{B.13}$$

Applying Itô's formula to $e^{-\rho t}[\tilde{q}_i,\tilde{v}_i]\check{s}$ and using (28) and (29), we obtain

$$\begin{aligned}
&-e^{-\rho T}E\big\{[\tilde{q}_i(T),\tilde{v}_i(T)]\check{s}(T)\big\}\\
=& E\int_0^T e^{-\rho t}\big[(c-\bar{p})\tilde{q}_i-\tilde{v}_i\bar{q}-2\tilde{u}_iB_i^T\check{s}\big]dt.
\end{aligned}$$

We have $\zeta_1=-\lim_{T\to\infty}\sum_{i=1}^N e^{-\rho T}E\big\{[\tilde{q}_i(T),\tilde{v}_i(T)]\check{s}(T)\big\}$. By (B.7) and (B.8),

$$e^{-\rho t}E|\tilde{q}_i(t)|^2=E\Big|\int_0^t e^{-(\mu+\frac{\rho}{2})(t-s)}e^{-\frac{\rho}{2}}\tilde{u}_i(s)ds\Big|^2=O(1),$$

which implies $E|\tilde{q}_i(t)|^2=O(e^{\rho t})$. By a similar argument, we have $E|\tilde{v}_i(t)|^2=O(e^{\rho t})$. Noting $\check{s}\in C_b([0,\infty),\mathbb{R}^2)$, we have $\zeta_1=0$. As in the proof of Lemma A.1 in Huang (2010), we use Jenson's inequality to get that

$$\begin{aligned}
&E\int_0^\infty e^{-\rho t}|q^{(N)}|^2dt\\
\le& C+CE\int_1^\infty e^{-\rho t}\Big[\int_0^t e^{-\mu(t-\tau)}\Big|\frac{1}{N}\sum_{i=1}^N b_iu_i(\tau)\Big|d\tau\Big]^2dt\\
\le& C+\frac{C}{\mu N(\rho+\mu)}E\int_1^\infty e^{-\rho t}\sum_{i=1}^N|u_i(t)|^2dt.
\end{aligned}\tag{B.14}$$

This together with (B.1) and (B.7) implies $E\int_0^\infty e^{-\rho t}|\tilde{q}^{(N)}|^2 dt < \infty$. By Schwarz's inequality and Lemma B.1, we have

$$\int_0^\infty e^{-\rho t}(\bar{p}-\check{p})\tilde{q}^{(N)}dt = O(\frac{1}{\sqrt{N}}+\varepsilon_N). \tag{B.15}$$

Furthermore, it follows from Schwarz's inequality, (B.7) and Lemma B.1 that $\int_0^\infty e^{-\rho t}\tilde{p}(\bar{q}-\check{q}^{(N)})dt = O(\frac{1}{\sqrt{N}}+\varepsilon_N)$. Thus, by (B.13) and (B.15) we have $|I^{(N)}| = |\zeta_2| = O(\frac{1}{\sqrt{N}}+\varepsilon_N)$. This completes the proof. □

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