MATH 2100 Assignment 13 Solutions

Problem 1. *Given a group G and element a with $|a| = 48$, find a divisor k of 48 such that*

a. $\langle a^{21}\rangle = \langle a^k\rangle$

b. $\langle a^{14}\rangle = \langle a^k\rangle$

c. $\langle a^{18}\rangle = \langle a^k\rangle$

Proof. All we need is Theorem 4.2:

$$\langle a^{21}\rangle = \langle a^{(48,21)}\rangle = \langle a^3\rangle$$
$$\langle a^{14}\rangle = \langle a^{(48,14)}\rangle = \langle a^2\rangle$$
$$\langle a^{18}\rangle = \langle a^{(48,18)}\rangle = \langle a^6\rangle$$

$\therefore$ The divisors k of 48 we need are 3, 2, and 6 respectively. $\square$

Problem 2. *Rewrite part (b) of problem 1, taking $G = \mathbb{Z}_{48}$ and $a = 1$ and using additive notation. Write out all elements of the equal groups.*

Proof. Changing notation does not change the answer, so we still have $k = 2$, i.e. $\langle 14(1)\rangle = \langle 2(1)\rangle$. The elements are as follows:

$$\langle 14\rangle = \{14, 28, 42, 8, 22, 36, 2, 16, 30, 44, 10, 24, 38, 4, 18, 32, 46, 12, 26, 40, 6, 20, 34, 0\}$$
$$\langle 2\rangle = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42, 44, 46, 0\}$$

$\square$

(**Note:** There are some interesting patterns to be found in the first arrangement. Take a look at where the multiples of 6 land, for instance.)

Problem 3. *Let G be a cyclic group with elements x, y such that $|x| = 16$, $|y| = 14$. Show that $x^8 = y^7$.*

Proof. The quickest solution is to notice the following:

$$(x^8)^2 = x^{16} = e$$
$$(y^7)^2 = y^{14} = e$$

Thus x^8 and y^7 have orders dividing 2. Moreover, we know $x^8 \neq e$ since $|x| = 16$, so x^8 must have order 2, and likewise y^7 has order 2.

Now $\langle x^8\rangle = \{e, x^8\}$ and $\langle y^7\rangle = \{e, y^7\}$ are both subgroups of G. But G is cyclic, so if it has an order-2 subgroup, it must be unique. Thus $\{e, x^8\} = \{e, y^7\}$; in particular, $x^8 \in \{e, y^7\}$. We have already seen that $x^8 \neq e$, so we must have $x^8 = y^7$. $\square$

(**Note 1:** We could also have done this by choosing a generator g for G, so that $x = g^k$, $y = g^l$. It's worth going through the proof this way to see how it works.)

(**Note 2:** An easy corollary of the above proof is that if a cyclic group has an order-2 element, it's unique. This is true of the order-1 element too. See if you can prove the *converse*: if a cyclic group has only one element of order n, then $n = 1$ or $n = 2$.)

Problem 4. *Simplify the permutations in Chapter 5, Exercise 5 and find their orders.*

Proof.

1. (124)(357) is already disjoint. Order: $[3, 3] = 3$.
2. (124)(3567) is disjoint. Order: $[3, 4] = 12$.
3. (124)(35) is disjoint. Order: $[3, 2] = 6$.
4. (124)(357869) is disjoint. Order: $[3, 6] = 6$.
5. (1235)(24567) = (124)(3567). Order: $[3, 4] = 12$.
6. (345)(245) = (25)(34). Order: $[2, 2] = 2$.

$\square$

(**Note:** The last one gives us a way to turn a product of disjoint 2-cycles into a product of 3-cycles. This actually turns out to be quite important – it's the first step in breaking down how the alternating group A_n works.)