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# On the numerical approximation of one-dimensional nonconservative hyperbolic systems

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ABSTRACT

Attempts to define weak solutions to nonconservative hyperbolic systems have lead to the development of several approaches, most notably the path-based theory of Dal Maso, LeFloch, and Murat (DLM) and the vanishing viscosity solutions described by Bianchini and Bressan. While these theories enable us to define weak solutions to nonconservative hyperbolic systems, difficulties arise when numerically approximating these systems. Specifically, in the neighborhood of a discontinuity, the numerical solutions tend to not converge to the theoretically specified weak solution of the system. This convergence error is easily seen in the numerical approximation of Riemann problems, in which the error appears and propagates at the formation of discontinuity waves. In this paper we investigate several methods to numerically approximate nonconservative hyperbolic systems, we discuss why these convergence errors arise, and by using recent results established by Alouges and Merlet we give an approximate description of what weak solutions these numerical solutions converge to. We then propose several strategies for the design of numerical schemes which reduce these convergence errors.


## 1. Introduction

The aim of this paper is to investigate the numerical approximation of nonconservative hyperbolic systems (NCHSs). In particular, we are interested in the construction of convergent numerical schemes for approximating such systems. Nonconservative hyperbolic systems arise in several areas of applied mathematics, in particular in the study of compressible multi-phase/fluid flows and have various industrial applications, such as two-phase flows in nuclear power plant reactors, solid rocket motors, chemical plants, detonations, shallow water bi-fluid flows, shallow water flows over irregular topography, and others [13,27,19,24,21]. These systems have proven to be difficult to analyse and have been much less studied than hyperbolic systems of conservation laws (see for instance [12]). Nevertheless, their wide range of applications has motivated large efforts to better understand these systems and their numerical approximations.

In this paper, we will be interested in the one dimensional NCHS,

$$\frac{\partial \mathbf{u}}{\partial t} + A(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x} = \mathbf{0}, \quad \mathbf{u} \in \Omega \subseteq \mathbb{R}^n, \ (x,t) \in \mathbb{R} \times \mathbb{R}^+, \tag{1.1}$$

where $\Omega$ is open convex set and $A$ is a smooth function $A : \mathbb{R}^n \to M_n(\mathbb{R})$. We assume that this system is strictly hyperbolic, that is, $A$ has $n$ real and distinct eigenvalues $\lambda_1(\mathbf{u}) < \lambda_2(\mathbf{u}) < \cdots < \lambda_n(\mathbf{u}), \forall \mathbf{u} \in \Omega$ with linearly independent eigenvectors. Recall that when $A$ is the Jacobian matrix of some vector-valued function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$, i.e. $A(\mathbf{u}) = D\mathbf{f}(\mathbf{u})$, then this system reduces to a hyperbolic system of conservation laws. For simplicity we will also assume that each characteristic field is genuinely nonlinear or linearly degenerate. For a genuinely nonlinear $k$-field, we also assume the normalization,

$$\nabla \lambda_k(\mathbf{u}) \cdot \mathbf{r}_k(\mathbf{u}) = 1, \quad \forall \mathbf{u} \in \Omega. \tag{1.2}$$

The intrinsic difficulty in studying NCHSs is how to the define shock wave solutions, namely the question of when

$$\mathbf{u}(x,t) = \begin{cases} \mathbf{u}_L, & x < \sigma t, \\ \mathbf{u}_R, & x > \sigma t. \end{cases}$$

is a weak solution of the nonconservative system. It is well-know that for systems of conservation laws $\mathbf{u}_L$, $\mathbf{u}_R$, and $\sigma$ must satisfy the Rankine–Hugoniot jump condition:

$$\sigma(\mathbf{u}_R - \mathbf{u}_L) = \mathbf{f}(\mathbf{u}_R) - \mathbf{f}(\mathbf{u}_L).$$

However, in the nonconservative case when such a function $\mathbf{f}(\mathbf{u})$ does not exist, we have no such condition. Furthermore, because of the nonconservative product $A(\mathbf{u})\mathbf{u}_x$, and the fact that products

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of distributions are not well-defined [25], we cannot a priori rigorously define the notion of weak solutions for system (1.1). Although this constitutes an old problem, two main distinct ways of overcoming this issue[1] in the framework of NCHSs have been proposed. The first is due to Dal Maso, LeFloch and Murat (DLM) [11,17] in which the authors define the nonconservative product not as a distribution, but as a bounded Borel measure which depends on a family of paths, $\phi : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. This family of paths satisfies the following properties,

$$\phi(0; \mathbf{u}_L, \mathbf{u}_R) = \mathbf{u}_L, \qquad \phi(1; \mathbf{u}_L, \mathbf{u}_R) = \mathbf{u}_R,$$

for all $\mathbf{u}_L$, $\mathbf{u}_R$ in $\mathbb{R}^n$, and the nonconservative product, denoted by $[A(\mathbf{u})\mathbf{u}_x]_\phi$, is defined as

$$[A(\mathbf{u})\mathbf{u}_x]_\phi(B) = \int_B A(\mathbf{u})\mathbf{u}_x \, dx,$$

when $\mathbf{u}$ is continuous on a Borel set $B$, and by

$$[A(\mathbf{u})\mathbf{u}_x]_\phi((x_0, t_0)) = \int_0^1 A(\phi(s; \mathbf{u}_L, \mathbf{u}_R)) \frac{\partial \phi}{\partial s}(s; \mathbf{u}_L, \mathbf{u}_R) \, ds,$$

when $\mathbf{u}$ has a jump discontinuity and $\mathbf{u}_L$ and $\mathbf{u}_R$ are the left and right limits of the discontinuity, respectively. This definition of the nonconservative product allows us to rigorously define which discontinuity waves are weak solutions of the system, and together with an entropy condition we are able to establish the local existence and uniqueness of entropy weak solutions of (1.1). It is important to note that this definition of a nonconservative product only applies to functions which are piecewise differentiable with finite jump discontinuities, and not for general distributions. However, a priori these functions are sufficient to solve the Riemann problem for sufficiently small jumps $||\mathbf{u}_R - \mathbf{u}_L||$.

The implementation of DLM theory in a numerical scheme for approximating NCHSs has been investigated by several authors. Originally Toumi proposed a path-based approach [28,29] to build a Roe solver for NCHSs. Later other authors, in particular Parés and Castro [6,5,20], and Rhebergen et al. [23,22] have investigated other numerical methods (Godunov, WENO, Discontinuous Galerkin, etc.) for these systems which are based on DLM path-theory. The fundamental issue with the numerical schemes based on the DLM framework is that in the presence of discontinuities the numerical solution may not converge to the specified entropic weak solution. A consequence of this is that the numerical solutions of NCHSs appear to have a convergence error once discontinuities form. This convergence error was first demonstrated by Hou and LeFloch in [16] for conservative equations written in nonconservative form, and later shown to be a general issue for NCHSs by several other authors [7,18]. A particular example of this convergence error was given by Abgrall and Karni in [2].

Another approach for defining weak solutions to NCHSs was developed in the recent works of Bianchini and Bressan [4], and Alouges and Merlet [3] who partially generalized results of Bianchini and Bressan. In these technical works, the authors investigate the solutions of the following viscous system,

$$\mathbf{u}_t^\varepsilon + A(\mathbf{u}^\varepsilon)\mathbf{u}_x^\varepsilon = \varepsilon(B(\mathbf{u}^\varepsilon)\mathbf{u}_x^\varepsilon)_x, \tag{1.3}$$

where $B(\mathbf{u})$ is positive-definite viscosity matrix. This system is a parabolic regularization of the original system (1.1). In this approach, entropy weak solutions of system (1.1) are defined to be the vanishing viscosity solutions of the viscous system, i.e. entropy weak solutions to (1.1) are constructed as the limit of the solutions to this viscous system as $\varepsilon \to 0$. Note that, unlike in conservative case, the weak solutions will depend on the choice of the viscosity matrix, $B(\mathbf{u})$. In the particular case when $B(\mathbf{u}) \equiv I$, Bianchini and Bressan showed that these vanishing viscosity solutions are unique and, in particular, they describe the shock curves and viscous shock profiles associated to this specific viscosity matrix. These results were then generalized by Alouges and Merlet in [3] to the case where the viscosity matrix $B(\mathbf{u})$ commutes with $A(\mathbf{u})$. The authors also propose a definition of shock curves of nonconservative systems as solution of the following dynamical system

$$\begin{cases} (A(\mathbf{u}) - \sigma I)\dfrac{d\mathbf{u}}{d\sigma} = \mathbf{u} - \mathbf{u}_L, \\ \mathbf{u}(\lambda_i(\mathbf{u}_L)) = \mathbf{u}_L. \end{cases} \tag{1.4}$$

They prove that the shock curves given by this system agree to at least $O(|\sigma - \lambda_k(\mathbf{u}_L)|^3)$ with the exact shock curves of the vanishing viscosity solutions of (1.3) when $B(\mathbf{u})$ commutes with $A(\mathbf{u})$. This result gives us a simple way to approximate the exact entropy weak solution of the Riemann problem, associated to this class of viscosity matrices.

The obvious drawback of these formulations is that the definition of entropy weak solutions depend on some additional information, i.e. the choice of path, $\phi$, or the choice of viscosity matrix, $B(\mathbf{u})$. Because of this, it is difficult to know which weak solutions will be the correct, 'physically relevant' solutions. As LeFloch remarks in [17], an appropriate choice of path in the DLM theory would be a parametrization the viscous profiles, associated to an appropriate physical viscosity matrix. However, the question of how to determine the viscous profiles is made difficult since it involves finding bounded solutions of an ODE on an infinite domain (for a more complete discussion see [26]). One possible approach to this issue is to use another useful result regarding the shock curves defined by system (1.4). In their work, Alouges and Merlet proved that when the viscosity matrix $B(\mathbf{u})$ commutes with $A(\mathbf{u})$, these shock curves will agree with the viscous shock profiles up to the third order in the size of discontinuity. However, in general we are not guaranteed that the physical viscosity will commute with the matrix $A(\mathbf{u})$, and therefore we do not know if these shock curves are a good approximation of the viscous profiles.

The important observation is that, although we are not guaranteed that the physically relevant viscosity will commute with $A(\mathbf{u})$, for many numerical schemes the *numerical* viscosity does indeed commute with $A(\mathbf{u})$. We can therefore use the results of Alouges and Merlet to give an approximate description of what weak solutions these schemes converge to. Furthermore, this tells us that for many numerical schemes the choice of path $\phi$ will have only a minor affect on the numerical solution for sufficiently small jumps between cells. Indeed, when using the DLM path-theory in a discontinuous Galerkin scheme, Rhebergen et al. [23] reported that the choice of path did not significantly altered their numerical results. While these results do not aid us in determining what solutions are physically relevant to the particular problem, it is an important theoretical step in understanding the nature of the numerical approximations of these nonconservative systems.

The remainder of this paper is organized as follows. In Section 2, we briefly recall several key aspects of the theory developed by Dal Maso, LeFloch and Murat, and we present the approximate shock curves of Alouges and Merlet. We then present a property of the Alouges–Merlet curves which we call reversibility. We then describe how the well-known Lax–Wendroff theorem fails for nonconservative systems, and how the convergence errors of the nonconservative schemes arise. The discussion motivates us to consider several schemes which implement the Alouges–Merlet shock curves directly, or use some approximation. By using the results of Alouges and Merlet, we propose several strategies for designing

[1] A third one is based on Colombeau's generalized functions [10], but is not discussed here.

schemes whose numerical solutions agree closely with the approximate vanishing viscosity solutions of system (1.3). We then apply some of these numerical schemes to a particular nonconservative system in Section 5.

## 2. Shock curves in non-conservative hyperbolic systems

In this section we recall some important features of nonconservative hyperbolic systems. As mentioned in the introduction, the main difficulties that we must address from the continuous point of view are how to define weak solutions to these systems, and how to properly define the shock curves in order to solve Riemann problems. Additional difficulties will appear in the numerical framework.

We begin this section by briefly recalling some important aspects of the path-dependent theory of nonconservative systems proposed by Dal Maso, LeFloch, and Murat (DLM). We will then present the definition of shock curves for nonconservative systems proposed by Alouges and Merlet. In particular, we will establish a local parametrization for these curves for weak shocks. We will also investigate a property of these shock curves known as *reversibility*, and establish that the Alouges–Merlet shock curves are indeed reversible. This property will be key later when we investigate the equivalent equations of several numerical schemes, see also [9].

We will then show how the ideas proposed in Alouges and Merlet's approach can be applied in the framework of Dal Maso, LeFloch, and Murat's path-theory. Specifically, we will derive a similar local parametrization of the DLM shock curves for weak shocks, and show that, in general, the DLM shock curves are not reversible for an arbitrary choice of path.

### 2.1. Dal Maso–LeFloch–Murat path theory

As a product of distributions $A(\mathbf{u})\mathbf{u}_x$, it is not clear how this term should be defined, and thus we are unable to specify what discontinuity waves can be weak solutions. The idea proposed by Dal Maso, LeFloch, and Murat [11] was to regard this term not as a distribution, but as a bounded Borel measure. Let us briefly recall the principle: we first introduce a family of paths, $\phi : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, satisfying the following properties:

1. $\forall \mathbf{u}_L, \mathbf{u}_R \in \mathbb{R}^n$,

$$\phi(0; \mathbf{u}_L, \mathbf{u}_R) = \mathbf{u}_L, \qquad \phi(1; \mathbf{u}_L, \mathbf{u}_R) = \mathbf{u}_R.$$

2. $\exists k > 0$, such that $\forall \mathbf{u}_L, \mathbf{u}_R \in \mathbb{R}^n$, $\forall s \in [0, 1]$,

$$\left| \frac{\partial \phi}{\partial s}(s; \mathbf{u}_L, \mathbf{u}_R) \right| \leqslant k|\mathbf{u}_L - \mathbf{u}_R|.$$

3. $\exists k > 0$, such that $\forall \mathbf{u}_L, \mathbf{u}_R, \mathbf{v}_L, \mathbf{v}_R \in \mathbb{R}^n$, $\forall s \in [0, 1]$,

$$\left| \frac{\partial \phi}{\partial s}(s; \mathbf{u}_L, \mathbf{u}_R) - \frac{\partial \phi}{\partial s}(s; \mathbf{v}_L, \mathbf{v}_R) \right| \leqslant k(|\mathbf{u}_L - \mathbf{u}_R| + |\mathbf{v}_L - \mathbf{v}_R|).$$

Then we define the nonconservative product, $A(\mathbf{u})\mathbf{u}_x$, as a bounded Borel measure denoted by $[A(\mathbf{u})\mathbf{u}_x]_\phi$, with the property that when $\mathbf{u}$ is smooth on a Borel set $B$, this measure is defined by

$$[A(\mathbf{u})\mathbf{u}_x]_\phi(B) = \int_B A(\mathbf{u})\mathbf{u}_x \, dx,$$

and if $\mathbf{u}$ is discontinuous at the point $x_0$ then,

$$[A(\mathbf{u})\mathbf{u}_x]_\phi(\{x_0\}) = \int_0^1 A(\phi(s; \mathbf{u}(x_0^-), \mathbf{u}(x_0^+))) \frac{\partial \phi}{\partial s}(s; \mathbf{u}(x_0^-), \mathbf{u}(x_0^+)) \, ds.$$

Using this definition for the nonconservative product it is possible to show [17] that we can generalize the Rankine–Hugoniot jump condition to:

$$\sigma(\mathbf{u}_R - \mathbf{u}_L) = \int_0^1 A(\phi(s; \mathbf{u}_L, \mathbf{u}_R)) \frac{\partial \phi}{\partial s}(s; \mathbf{u}_L, \mathbf{u}_R) \, ds. \tag{2.1}$$

In this paper, we will refer to this generalized jump condition as the DLM jump condition. Finally, once we have this jump condition for some chosen path, we can define the $k$-shock curves for the genuinely nonlinear $k$-fields and proceed to solve the Riemann problem as in the conservative case. It is clear, however, that different choices of the path $\phi$ will lead to different solutions of the Riemann problem, and the question of which choice of path will yield physical, entropy solutions is far from trivial. To see this, let us consider the vanishing viscosity entropy condition by introducing an admissible viscosity matrix $B(\mathbf{u})$ (smooth and positive) to system (1.1):

$$\mathbf{u}_t^\varepsilon + A(\mathbf{u}^\varepsilon)\mathbf{u}_x^\varepsilon = \varepsilon(B(\mathbf{u}^\varepsilon)\mathbf{u}_x^\varepsilon)_x.$$

We can then look at the viscous profiles $\mathbf{u}^\varepsilon(x, t) = \mathbf{v}((x - \sigma t)/\varepsilon)$. The resulting ODE is

$$(A(\mathbf{v}) - \sigma)\mathbf{v}' = (B(\mathbf{v})\mathbf{v}')'.$$

Next, let us suppose the formal vanishing viscosity limit of this viscous profile is a shock wave, i.e.

$$\lim_{\varepsilon \to 0} \mathbf{u}^\varepsilon(x, t) = \begin{cases} \mathbf{u}_L, & x < \sigma t, \\ \mathbf{u}_R, & x > \sigma t. \end{cases}$$

Then the viscous profile will have the form

$$\mathbf{u}^\varepsilon(x, t) = \begin{cases} \mathbf{u}_L, & x < \sigma t - \varepsilon, \\ \nu_B\left(\dfrac{x - \sigma t + \varepsilon}{2\varepsilon}\right), & \sigma t - \varepsilon \leqslant x \leqslant \sigma t + \varepsilon, \\ \mathbf{u}_R, & x > \sigma t + \varepsilon, \end{cases}$$

where $\nu_B$ is a smooth function with the properties $\nu_B(0) = \mathbf{u}_L$ and $\nu_B(1) = \mathbf{u}_R$. Considering $\mathbf{u}^\varepsilon$ as a measure we see that

$$\lim_{\varepsilon \to 0} [A(\mathbf{u}^\varepsilon)\mathbf{u}_x^\varepsilon] = \left( \int_0^1 A(\nu_B(s)) \frac{\partial \nu_B}{\partial s} \, ds \right) \delta_{x - \sigma t},$$

with the convergence in the sense of measures. Thus, in order to obtain the vanishing viscosity solution relative to the viscosity $B(\mathbf{u})$, we must choose our path, $\phi$ to be precisely the viscous profile $\nu_B$.[2] It is this fact that introduces difficulty when designing a numerical scheme using DLM path-theory. Although we may choose some path $\phi$ to define our shock curves, and therefore our 'exact' solution to the Riemann problem, the numerical scheme which we use will contain some numerical viscosity and the numerical solution will converge to the vanishing viscosity limit relative to this numerical viscosity. The fundamental problem is that the path $\phi$ is usually not chosen to be a parametrization of the viscous profiles for this numerical viscosity. In fact, it would be very difficult to do so. It is this inconsistency between the choice of $\phi$ and the chosen

[2] Notice that, as in the conservative case, the viscous profiles will, in general, depend on the viscosity matrix, $B$. However, in the conservative case the shock curves are defined using only the Rankine–Hugoniot jump condition and thus do not depend on the choice of $B$.

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'exact' solution and the solution which the numerical scheme converges to, which produces the convergence error which has been observed by numerous authors [1,7,16,18]. However, as we will explain below, the shock curves defined by Alouges and Merlet have the property that they agree with the viscous profiles of a nonconservative system for a general class of viscosity matrices, up to the third order near a given left state. Therefore, these shock curves give us a way to approximate not only the shock curves of a nonconservative system for a more general viscosity, but also the viscous profiles.

### 2.2. Alouges–Merlet approximate shock curves

The motivation for the definition of approximate shock curves proposed by Alouges and Merlet stems from the fact that they accurately approximate the shock curves for NCHSs which are found by a vanishing viscosity process. These vanishing viscosity solutions were extensively studied by Bianchini and Bressan in [4]. In their paper, the authors define solutions to general nonconservative systems by considering a regularization of the system (1.1),

$$\mathbf{u}_t^\varepsilon + A(\mathbf{u}^\varepsilon)\mathbf{u}_x^\varepsilon = \varepsilon(B(\mathbf{u}^\varepsilon)\mathbf{u}_x^\varepsilon)_x. \tag{2.2}$$

More specifically, they consider the case where the viscosity matrix $B(\mathbf{u})$ is the identity matrix, $I(\mathbf{u})$. Using a regularization technique, the authors define solutions to (1.1) as the unique limits of solutions to this viscous system as $\varepsilon \to 0$. The details are beyond the scope of this paper, but we will state their main results. For a very general $A(\mathbf{u})$ (no genuine nonlinearity assumptions, etc.), the authors solve the Riemann problem for $\mathbf{u}_L$ and $\mathbf{u}_R$ sufficiently close and recover the classical succession of self-similar $k$-waves and characterize them.

The obvious shortcoming of this study is that vanishing viscosity solutions of this system for a more general viscosity matrix are not established by this theory. However, in the paper by Alouges and Merlet [3] these results are extended to the case where the admissible viscosity matrix $B(\mathbf{u})$ is assumed to commute with $A(\mathbf{u})$. The authors establish the same results as Bianchini and Bressan in this more general setting.

Alouges and Merlet also propose a new definition for shock curves in the nonconservative case. Suppose for the moment that the hyperbolic system we are considering is conservative, and consider an admissible $k$-shock wave with left state $\mathbf{u}_L$, and right state $\mathbf{u}_R$, propagating with speed $\sigma$. If we consider $\mathbf{u}_R$ as a function of $\sigma$, the Rankine–Hugoniot jump condition writes

$$\mathbf{f}(\mathbf{u}_R(\sigma)) - \mathbf{f}(\mathbf{u}_L) = \sigma(\mathbf{u}_R(\sigma) - \mathbf{u}_L),$$

with $\mathbf{u}_R(\lambda_k(\mathbf{u}_L)) = \mathbf{u}_L$. Differentiating this with respect to $\sigma$ yields

$$\begin{cases} (A(\mathbf{u}_R) - \sigma I)\dfrac{d\mathbf{u}_R}{d\sigma} = \mathbf{u}_R - \mathbf{u}_L, \\ \mathbf{u}_R(\lambda_k(\mathbf{u}_L)) = \mathbf{u}_L. \end{cases} \tag{2.3}$$

Alouges and Merlet use this system to define approximate shock curves for nonconservative systems.

**Definition 2.1** *(Alouges–Merlet Shock curves [3]).* We call a non-constant solution of (2.3) an Alouges–Merlet shock curve of the nonconservative system (1.1).

An obviously trivial solution to this differential equation is $\mathbf{u}_R(\sigma) \equiv \mathbf{u}_L$. Notice that this differential equation is not classical since there is a degeneracy at the initial point $\mathbf{u}_R(\lambda_k(\mathbf{u}_L)) = \mathbf{u}_L$, for each $k$. To overcome this, the authors prove [3] the following result.

**Proposition 2.1.** *Suppose that the k-th field is genuinely nonlinear with normalization (1.2). Then Eq. (2.3) has a unique, non-trivial solution in the neighborhood of $\lambda_k(\mathbf{u}_L)$. Moreover, the non-trivial solution satisfies*

$$\mathbf{u}_R(\sigma) = \mathbf{u}_L + 2(\sigma - \lambda_k(\mathbf{u}_L))\mathbf{r}_k(\mathbf{u}_L) + O(|\sigma - \lambda_k(\mathbf{u}_L)|^2).$$

The degeneracy of (2.3) is thus overcome by adding the initial condition

$$\frac{d\mathbf{u}_R}{d\sigma}(\lambda_k(\mathbf{u}_L)) = 2\mathbf{r}_k(\mathbf{u}_L), \tag{2.4}$$

to the differential equation. In fact, we can extend the above result to include the second order terms which gives a more precise description of these shock waves.

**Proposition 2.2.** *For a genuinely nonlinear k-field, the unique and non-trivial solution to (2.3) satisfies:*

$$\mathbf{u}_R(\sigma) = \mathbf{u}_L + 2(\sigma - \lambda_k(\mathbf{u}_L))\mathbf{r}_k(\mathbf{u}_L) + 2(\sigma - \lambda_k(\mathbf{u}_L))^2 D\mathbf{r}_k(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L) + O(|\sigma - \lambda_k(\mathbf{u}_L)|^3).$$

**Proof.** We can prove this proposition by simply performing a Taylor expansion of $\mathbf{u}_R(\sigma)$ around $\lambda_k(\mathbf{u}_L)$ and using the differential equation to find the higher order term. Let us expand $\mathbf{u}_R(\sigma)$ as

$$\mathbf{u}_R(\sigma) = \mathbf{u}_L + 2(\sigma - \lambda_k(\mathbf{u}_L))\mathbf{r}_k(\mathbf{u}_L) + \frac{1}{2}R(\mathbf{u}_L)(\sigma - \lambda_k(\mathbf{u}_L))^2 + O(|\sigma - \lambda_k(\mathbf{u}_L)|^3),$$

where $R(\mathbf{u}_L) = (d^2\mathbf{u}_R/d\sigma^2)(\mathbf{u}_L)$ is to be determined. Using this expression, we can expand the solution of the system (2.3) around $\sigma = \lambda_k(\mathbf{u}_L)$ and after calculation we obtain:

$$4[DA(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L) - I]\mathbf{r}_k(\mathbf{u}_L) + [A(\mathbf{u}_L) - \lambda_k(\mathbf{u}_L)I]R(\mathbf{u}_L) = \mathbf{0}. \tag{2.5}$$

In order to determine $R(\mathbf{u}_L)$, let us consider the identity

$$[A(\mathbf{u}_R(\sigma)) - \lambda_k(\mathbf{u}_R(\sigma))I]\mathbf{r}_k(\mathbf{u}_R(\sigma)) = \mathbf{0},$$

and differentiate with respect to $\sigma$ to obtain:

$$\left[DA(\mathbf{u}_R(\sigma))\frac{d\mathbf{u}_R}{d\sigma} - \left(\nabla\lambda_k(\mathbf{u}_R(\sigma))\cdot\frac{d\mathbf{u}_R}{d\sigma}\right)I\right]\mathbf{r}_k(\mathbf{u}_R(\sigma)) + [A(\mathbf{u}_R(\sigma)) - \lambda_k(\mathbf{u}_R(\sigma))I]D\mathbf{r}_k(\mathbf{u}_R(\sigma))\cdot\frac{d\mathbf{u}_R}{d\sigma} = 0.$$

Evaluating this at $\sigma = \lambda_k(\mathbf{u}_L)$, and using the initial conditions (2.3) and (2.4), we obtain:

$$2[DA(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L) - I]\mathbf{r}_k(\mathbf{u}_L) + 2[A(\mathbf{u}_L) - \lambda_k(\mathbf{u}_L)I]D\mathbf{r}_k(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L) = 0.$$

Comparing this expression with (2.5), we find that

$$R(\mathbf{u}_L) = \frac{d^2\mathbf{u}_R}{d\sigma^2}(\mathbf{u}_L) = 4D\mathbf{r}_k(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L).$$

which completes the proof. □

**Remark 2.1.** For a genuinely nonlinear $k$-field with normalization (1.2), we can also express the local expansion as:

$$\mathbf{u}_R = \mathbf{u}_L + \varepsilon\mathbf{r}_k(\mathbf{u}_L) + \frac{1}{2}\varepsilon^2 D\mathbf{r}_k(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L) + O(\varepsilon^3), \tag{2.6}$$

$$\sigma = \lambda_k(\mathbf{u}_L) + \frac{1}{2}\varepsilon + O(\varepsilon^2), \tag{2.7}$$

where $\varepsilon = \lambda_k(\mathbf{u}_R) - \lambda_k(\mathbf{u}_L)$. Notice that (2.6) is the same local expansion as for the $k$-rarefaction curves. This expansion will be useful in the sections below.

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It is clear that in the conservative case, when $A(\mathbf{u})$ is a Jacobian matrix, this shock curve definition will recover the correct curves. Alouges and Merlet prove that these approximate shock curves agree with the ones found by the vanishing viscosity process of Bianchini and Bressan (and defined in [3]) up to the third order near a given left state. Thus (2.3) gives us a simple way to approximate the shock curves as described by Bianchini and Bressan, i.e. the exact vanishing viscosity solutions of the Riemann problem in the nonconservative case relative to the viscosity matrix, $B(\mathbf{u})$, which commutes with $A(\mathbf{u})$. Moreover, since (2.3) is independent of $B(\mathbf{u})$, the approximate solutions are also $B$-independent.

Another point of interest is that these approximate shock curves coincide with the viscous shock profiles to the third order near a given left state. To see this, let us consider the viscous system (2.2) and let us examine the $k$-shock profiles, which have the form $\mathbf{u}^\varepsilon(x, t) = U((x-\sigma t)/\varepsilon\,;\sigma) = U(\xi\,;\sigma)$. Then these shock profiles will solve the following system,

$$\begin{cases} (A(U)-\sigma I)U_\xi = (B(U)U_\xi)_\xi, \\ U(-\infty;\sigma) = \mathbf{u}_L, \quad \forall\sigma, \\ U(\xi;\lambda_k(\mathbf{u}_L)) \equiv \mathbf{u}_L. \end{cases}$$

The $k$-shock curve, $\mathcal{S}_k(\mathbf{u}_L)$, is defined by these profiles by $\mathbf{u}_R(\sigma) = U(+\infty\,;\sigma)$. Integrating this system along the profiles gives

$$\sigma(\mathbf{u}_R(\sigma)-\mathbf{u}_L) = \int_{\mathbb{R}} A(U)U_\xi\, d\xi,$$

and differentiating with respect to $\sigma$ and integrating by parts we obtain

$$(A(\mathbf{u}_R(\sigma))-\sigma I)\frac{d\mathbf{u}_R}{d\sigma} = \mathbf{u}_R(\sigma)-\mathbf{u}_L + \int_{\mathbb{R}} A(U)_\xi U_\sigma - A(U)_\sigma U_\xi\, d\xi.$$

So that we recover system (2.3) up to the term

$$R(U,\sigma) = \int_{\mathbb{R}} A(U)_\xi U_\sigma - A(U)_\sigma U_\xi\, d\xi.$$

Now we note that if $A$ is indeed a Jacobian matrix then $R(U, \sigma)$ vanishes. Also, if the shock curves and the shock profiles coincide then $R(U, \sigma)$ will again vanish. As noted earlier, approximate shock curves defined by (2.3) do indeed recover the correct shock curves up to the third order so $R(U, \sigma) = O(|\sigma-\lambda_k(\mathbf{u}_L)|^3)$. This tells us that these approximate shock curves are in fact also close to viscous shock profiles. As stated above, this results is interesting since it gives us a way to approximate the viscous shock profiles which is a piece of information useful in DLM path-theory.

### 2.3. Reversibility of Alouges–Merlet shock curves

In this section we introduce a new property of shock curves which we call *reversibility*. While it is not immediately obvious why this property is important, we will see in the sections below that this is a useful property when investigating the convergence of numerical schemes. Before we present this definition let us recall what it means for a state to be "connected" to another by a shock wave.

**Definition 2.2.** We say that a state $\mathbf{u}_R$ can be connected to a state $\mathbf{u}_L$ on the left by a $k$-shock wave if $\mathbf{u}_R$ lies on the $k$-shock curve of $\mathbf{u}_L$, denoted $\mathcal{S}_k(\mathbf{u}_L)$, for some speed $\sigma$. Similarly, we say that a state $\mathbf{u}_L$ can be connected to a state $\mathbf{u}_R$ on the right by a $k$-shock wave if $\mathbf{u}_L$ lies on the $k$-shock curve[3] of $\mathbf{u}_R$, $\mathcal{S}_k(\mathbf{u}_R)$, for some speed $\sigma$.

[3] Recall that to connect a state on the right, we consider the non-entropy part of the $k$-shock curve.

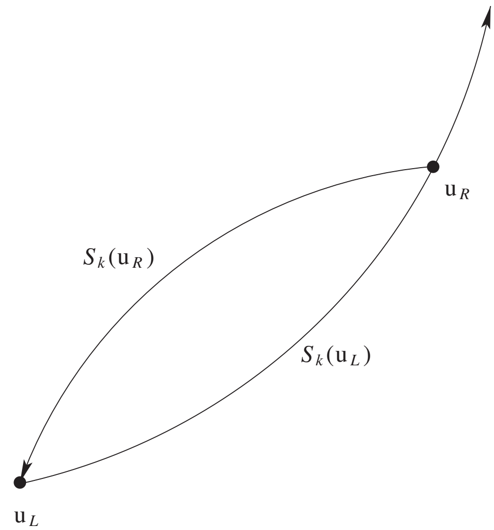


**Fig. 1.** Depiction of reversible shock curves.

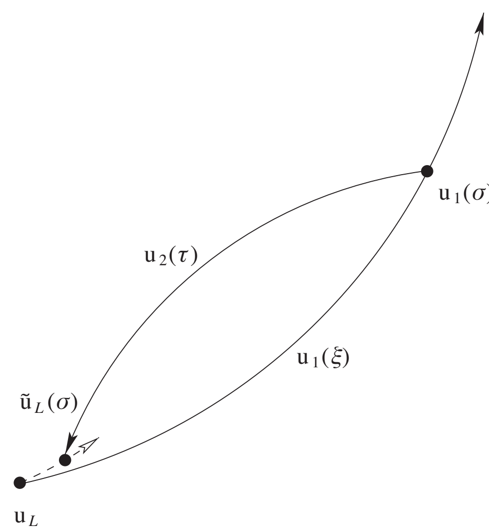


**Fig. 2.** In general, the DLM shock curves will not be reversible.

Now let us present the definition of reversible shock curves.

**Definition 2.3.** A $k$-shock curve for a nonconservative hyperbolic system is said to be reversible if for any right state $\mathbf{u}_R$ which can be connected to a left state $\mathbf{u}_L$ on the left by a $k$-shock wave traveling with speed $\sigma$, then $\mathbf{u}_L$ can be connected to the state $\mathbf{u}_R$ on the right by a $k$-shock wave traveling with speed $\sigma$.

Recall that in the conservative case when a state $\mathbf{u}_R$ lies on a $k$-shock curve of a state $\mathbf{u}_L$, i.e. $\mathbf{u}_R \in \mathcal{S}_k(\mathbf{u}_L)$, then from the Rankine–Hugoniot jump condition we know immediately that $\mathbf{u}_L$ will lie on the $k$-shock curve of the state $\mathbf{u}_R$ and hence shock curves in the conservative case are clearly reversible. In their paper, Alouges and Merlet point out that it is unclear whether this property will hold in the nonconservative case using their shock curves. Here, we will show this is indeed true (Figs. 1 and 2).

**Theorem 2.1.** *Assume the kth characteristic field is genuinely nonlinear, and let $\mathcal{S}_k(\mathbf{u}_L)$ be the Alouges–Merlet k-shock curve of the state $\mathbf{u}_L$ for the system (1.1), defined by (2.3). Suppose that $\mathbf{u}_R \in \mathcal{S}_k(\mathbf{u}_L) \cap \mathcal{V}(\mathbf{u}_L)$, where $\mathcal{V}(\mathbf{u}_L)$ is a neighborhood of $\mathbf{u}_L$, that is a state $\mathbf{u}_R$ in a neighborhood of $\mathbf{u}_L$ can be connected to $\mathbf{u}_L$ on the left by a k-shock wave traveling with speed $\sigma$. Furthermore, suppose $\tilde{\mathbf{u}}_L(\sigma) \in \mathcal{S}_k(\mathbf{u}_R) \cap \mathcal{V}(\mathbf{u}_R)$*

*is a state which can be connected to* $\mathbf{u}_R$ *on the right by a k-shock wave, again traveling with speed* $\sigma$. *Then* $\tilde{\mathbf{u}}_L(\sigma) \equiv \mathbf{u}_L$, $\forall\sigma$, *i.e. the Alouges–Merlet approximate shock curves are reversible.*

In order to prove this theorem we will need to make use of the following lemma.

**Lemma 2.1.** *Let* $\mathbf{u}_L$ *and* $\tilde{\mathbf{u}}_L$ *be defined as in Theorem 2.1. Then* $\tilde{\mathbf{u}}_L = \mathbf{u}_L + O(|\sigma - \lambda_k(\mathbf{u}_L)|^2)$. *In particular* $d\tilde{\mathbf{u}}_L/d\sigma(\lambda_k(\mathbf{u}_L)) = \mathbf{0}$.

**Proof.** This lemma is easily proved using a Taylor expansion of $\mathbf{u}_R$ and $\tilde{\mathbf{u}}_L$. By Proposition 2.2, we have the following expansions,

$$\mathbf{u}_R = \mathbf{u}_L + 2(\sigma - \lambda_k(\mathbf{u}_L))\mathbf{r}_k(\mathbf{u}_L) + O(|\sigma - \lambda_k(\mathbf{u}_L)|^2), \tag{2.8}$$

$$\tilde{\mathbf{u}}_L = \mathbf{u}_R + 2(\sigma - \lambda_k(\mathbf{u}_R))\mathbf{r}_k(\mathbf{u}_R) + O(|\sigma - \lambda_k(\mathbf{u}_R)|^2). \tag{2.9}$$

Using (2.8) with the normalization (1.2) we can expand $\sigma - \lambda_k(\mathbf{u}_R)$ as,

$$\begin{aligned}\sigma - \lambda_k(\mathbf{u}_R) &= \sigma - \lambda_k(\mathbf{u}_L + 2(\sigma - \lambda_k(\mathbf{u}_L))\mathbf{r}_k(\mathbf{u}_L) + O(|\sigma - \lambda_k(\mathbf{u}_L)|^2)),\\ &= \sigma - \lambda_k(\mathbf{u}_L) - 2(\sigma - \lambda_k(\mathbf{u}_L)) + O(|\sigma - \lambda_k(\mathbf{u}_L)|^2)),\\ &= -(\sigma - \lambda_k(\mathbf{u}_L)) + O(|\sigma - \lambda_k(\mathbf{u}_L)|^2).\end{aligned}$$

Thus, using this and (2.8) in (2.9), we can write $\tilde{\mathbf{u}}_L$ as,

$$\begin{aligned}\tilde{\mathbf{u}}_L &= \mathbf{u}_L + 2(\sigma - \lambda_k(\mathbf{u}_L))\mathbf{r}_k(\mathbf{u}_L)\\ &\quad -2(\sigma - \lambda_k(\mathbf{u}_L))\mathbf{r}_k(\mathbf{u}_L + O(|\sigma - \lambda_k(\mathbf{u}_L)|)) + O(|\sigma - \lambda_k(\mathbf{u}_L)|^2),\\ &= \mathbf{u}_L + O(|\sigma - \lambda_k(\mathbf{u}_L)|^2),\end{aligned}$$

which completes the proof. □

Before presenting a formal proof of the theorem, let us give a brief outline. Since $\tilde{\mathbf{u}}_L$ is defined by the composition of two curves, which are defined using the differential Eq. (2.3), we will first find the differential equation which governs $\tilde{\mathbf{u}}_L$. We then show that $\tilde{\mathbf{u}}_L(\sigma) \equiv \mathbf{u}_L$ is the only possible solution to this differential equation. In order to accomplish this, we will note the governing equation for $\tilde{\mathbf{u}}$ is of a similar form as the Alouges–Merlet shock system (2.3). We can then use Lemma 2.1 to conclude that $\tilde{\mathbf{u}}_L(\sigma)$ has the initial condition $d\tilde{\mathbf{u}}_L/d\sigma(\lambda_k(\mathbf{u}_L)) = \mathbf{0}$. But this initial condition will guarantee us that $\tilde{\mathbf{u}}_L(\sigma)$ is trivial, hence $\tilde{\mathbf{u}}_L(\sigma) \equiv \mathbf{u}_L$.

**Proof** *(Proof of Theorem).* We first wish to derive the governing differential equation for $\tilde{\mathbf{u}}_L$. Let us consider a genuinely nonlinear $k$-th field and consider the approximate $k$-shock curve of a fixed left state. Let us denote this curve by $\mathbf{u}_1(\xi)$. Then $\mathbf{u}_1(\xi)$ satisfies

$$\begin{cases}(A(\mathbf{u}_1) - \xi I)\dfrac{d\mathbf{u}_1}{d\xi} = \mathbf{u}_1 - \mathbf{u}_L,\\ \mathbf{u}_1(\lambda_k(\mathbf{u}_L)) = \mathbf{u}_L.\end{cases} \tag{2.10}$$

Let us select a point on this curve, say $\mathbf{u}_1(\sigma)$, for some $\sigma$. We wish to determine the state $\tilde{\mathbf{u}}_L(\sigma)$ which will lie on the $k$-shock curve of $\mathbf{u}_1(\sigma)$, that we denote by $\mathbf{u}_2(\tau;\sigma)$. Then $\mathbf{u}_2(\tau;\sigma)$ satisfies

$$\begin{cases}(A(\mathbf{u}_2) - \tau I)\dfrac{\partial\mathbf{u}_2}{\partial\tau} = \mathbf{u}_2 - \mathbf{u}_1(\sigma),\\ \mathbf{u}_2(\lambda_k(\mathbf{u}_1(\sigma));\sigma) = \mathbf{u}_1(\sigma).\end{cases} \tag{2.11}$$

Let us integrate (2.10) from $\lambda_k(\mathbf{u}_L)$ to $\sigma$ to obtain

$$\int_{\lambda_k(\mathbf{u}_L)}^{\sigma} (A(\mathbf{u}_1) - \xi I)\frac{d\mathbf{u}_1}{d\xi}\,d\xi = \int_{\lambda_k(\mathbf{u}_L)}^{\sigma} \mathbf{u}_1(\xi) - \mathbf{u}_L\,d\xi,$$

$$\int_{\lambda_k(\mathbf{u}_L)}^{\sigma} A(\mathbf{u}_1)\frac{d\mathbf{u}_1}{d\xi}\,d\xi = \int_{\lambda_k(\mathbf{u}_L)}^{\sigma} \xi\frac{d\mathbf{u}_1}{d\xi} + \mathbf{u}_1(\xi) - \mathbf{u}_L\,d\xi,$$

$$\int_{\lambda_k(\mathbf{u}_L)}^{\sigma} A(\mathbf{u}_1)\frac{d\mathbf{u}_1}{d\xi}\,d\xi = \sigma(\mathbf{u}_1(\sigma) - \mathbf{u}_L).$$

Similarly, we integrate (2.11) from $\lambda_k(\mathbf{u}_1(\sigma))$ to $\sigma$ to obtain

$$\int_{\lambda_k(\mathbf{u}_1(\sigma))}^{\sigma} A(\mathbf{u}_2(\tau;\sigma))\frac{\partial\mathbf{u}_2}{\partial\tau}(\tau;\sigma)\,d\tau = \sigma(\mathbf{u}_2(\sigma;\sigma) - \mathbf{u}_1(\sigma)).$$

The point we are interested in is $\tilde{\mathbf{u}}_L(\sigma) = \mathbf{u}_2(\sigma;\sigma)$. Adding these two equations, we obtain

$$\int_{\lambda_k(\mathbf{u}_L)}^{\sigma} A(\mathbf{u}_1)\frac{d\mathbf{u}_1}{d\xi}\,d\xi + \int_{\lambda_k(\mathbf{u}_1(\sigma))}^{\sigma} A(\mathbf{u}_2)\frac{\partial\mathbf{u}_2}{\partial\tau}\,d\tau = \sigma(\tilde{\mathbf{u}}_L(\sigma) - \mathbf{u}_L).$$

Note that this entire expression depends on the parameter $\sigma$. Let us differentiate this equation with respect to $\sigma$, to obtain

$$\begin{aligned}&A(\mathbf{u}_1(\sigma))\frac{d\mathbf{u}_1}{d\xi}(\sigma) + A(\mathbf{u}_2(\sigma;\sigma))\frac{\partial\mathbf{u}_2}{\partial\tau}(\sigma;\sigma) - A(\mathbf{u}_2(\lambda_k(\mathbf{u}_1(\sigma));\sigma))\\ &\frac{\partial\mathbf{u}_2}{\partial\tau}(\lambda_k(\mathbf{u}_1(\sigma));\sigma)\left(\nabla\lambda_k(\mathbf{u}_1(\sigma))\cdot\frac{d\mathbf{u}_1}{d\xi}(\sigma)\right)\\ &+\int_{\lambda_k(\mathbf{u}_1(\sigma))}^{\sigma}\frac{d}{d\sigma}\left[A(\mathbf{u}_2)\frac{\partial\mathbf{u}_2}{\partial\tau}\right]d\tau = \tilde{\mathbf{u}}_L(\sigma) - \mathbf{u}_L + \sigma\frac{d\tilde{\mathbf{u}}_L}{d\sigma}(\sigma).\end{aligned}$$

Using the initial condition $\mathbf{u}_2(\lambda_k(\mathbf{u}_1(\sigma));\sigma) = \mathbf{u}_1(\sigma)$ and the fact that $\tilde{\mathbf{u}}_L(\sigma) = \mathbf{u}_2(\sigma;\sigma)$, this reads:

$$\begin{aligned}&A(\mathbf{u}_1(\sigma))\frac{d\mathbf{u}_1}{d\xi}(\sigma) + A(\tilde{\mathbf{u}}_L(\sigma))\frac{\partial\mathbf{u}_2}{\partial\tau}(\sigma;\sigma) - A(\mathbf{u}_1(\sigma))\frac{\partial\mathbf{u}_2}{\partial\tau}\\ &(\lambda_k(\mathbf{u}_1(\sigma));\sigma)\left(\nabla\lambda_k(\mathbf{u}_1(\sigma))\cdot\frac{d\mathbf{u}_1}{d\xi}(\sigma)\right) + \int_{\lambda_k(\mathbf{u}_1(\sigma))}^{\sigma}\frac{d}{d\sigma}\left[A(\mathbf{u}_2)\frac{\partial\mathbf{u}_2}{\partial\tau}\right]\\ &d\tau = \tilde{\mathbf{u}}_L(\sigma) - \mathbf{u}_L + \sigma\frac{d\tilde{\mathbf{u}}_L}{d\sigma}(\sigma).\end{aligned} \tag{2.12}$$

Let us first examine the integral term in this expression. Expanding and integrating by parts we obtain,

$$\begin{aligned}&\int_{\lambda_k(\mathbf{u}_1(\sigma))}^{\sigma}\frac{d}{d\sigma}\left[A(\mathbf{u}_2)\frac{\partial\mathbf{u}_2}{\partial\tau}\right]d\tau\\ &= \int_{\lambda_k(\mathbf{u}_1(\sigma))}^{\sigma} A(\mathbf{u}_2)_\sigma\frac{\partial\mathbf{u}_2}{\partial\tau} + A(\mathbf{u}_2)\frac{\partial^2\mathbf{u}_2}{\partial\tau\partial\sigma}\,d\tau\\ &= A(\mathbf{u}_2(\sigma;\sigma))\frac{\partial\mathbf{u}_2}{\partial\sigma}(\sigma;\sigma) - A(\mathbf{u}_2(\lambda_k(\mathbf{u}_1(\sigma));\sigma))\\ &\quad\frac{\partial\mathbf{u}_2}{\partial\sigma}(\lambda_k(\mathbf{u}_1(\sigma));\sigma) + \int_{\lambda_k(\mathbf{u}_1(\sigma))}^{\sigma} A(\mathbf{u}_2)_\sigma\frac{\partial\mathbf{u}_2}{\partial\tau} - A(\mathbf{u}_2)_\tau\frac{\partial\mathbf{u}_2}{\partial\sigma}\,d\tau\\ &= A(\tilde{\mathbf{u}}_L(\sigma))\frac{\partial\mathbf{u}_2}{\partial\sigma}(\sigma;\sigma) - A(\mathbf{u}_1(\sigma))\frac{\partial\mathbf{u}_2}{\partial\sigma}(\lambda_k(\mathbf{u}_1(\sigma));\sigma) + R(\mathbf{u}_2,\sigma).\end{aligned}$$

Here we used the notation $R(\mathbf{u}_2,\sigma) = \int_{\lambda_k(\mathbf{u}_1(\sigma))}^{\sigma} A(\mathbf{u}_2)_\sigma\frac{\partial\mathbf{u}_2}{\partial\tau} - A(\mathbf{u}_2)_\tau\frac{\partial\mathbf{u}_2}{\partial\sigma}\,d\tau$. Inserting this expression into (2.12), we obtain,

$$\begin{aligned}&A(\tilde{\mathbf{u}}_L(\sigma))\left(\frac{\partial\mathbf{u}_2}{\partial\tau}(\sigma;\sigma) + \frac{\partial\mathbf{u}_2}{\partial\sigma}(\sigma;\sigma)\right) + A(\mathbf{u}_1(\sigma))\\ &\times\left[\frac{d\mathbf{u}_1}{d\xi}(\sigma) - \frac{\partial\mathbf{u}_2}{\partial\tau}(\lambda_k(\mathbf{u}_1(\sigma));\sigma)\cdot\left(\nabla\lambda_k(\mathbf{u}_1(\sigma))\cdot\frac{d\mathbf{u}_1}{d\xi}(\sigma)\right)\right.\\ &\left.-\frac{\partial\mathbf{u}_2}{\partial\sigma}(\lambda_k(\mathbf{u}_1(\sigma));\sigma)\right] + R(\mathbf{u}_2,\sigma) = \tilde{\mathbf{u}}_L(\sigma) - \mathbf{u}_L + \sigma\frac{d\tilde{\mathbf{u}}_L}{d\sigma}(\sigma).\end{aligned}$$

Finally, note that $(d/d\sigma)\tilde{\mathbf{u}}_L = (\partial\mathbf{u}_2/\partial\tau)(\sigma;\sigma) + (\partial\mathbf{u}_2/\partial\sigma)(\sigma;\sigma)$ and notice that the term within the square braces it simply the derivative of the initial condition for $\mathbf{u}_2(\tau)$. Thus, this term vanishes and we obtain,

$$(A(\tilde{\mathbf{u}}_L)-\sigma I)\frac{d}{d\sigma}\tilde{\mathbf{u}}_L + R(\mathbf{u}_2,\sigma) = \tilde{\mathbf{u}}_L(\sigma)-\mathbf{u}_L. \tag{2.13}$$

Notice that this system is similar to the shock curve system proposed by Alouges and Merlet, the difference being the additional term $R(\mathbf{u}_2, \sigma)$. Since clearly $R(\mathbf{u}_2, \sigma)$ is $O(|\sigma-\lambda_k(\mathbf{u}_L)|)$, this system suffers from the same degeneracy as the original systems studied by Alouges and Merlet. From Lemma 2.1 we see that the degeneracy must be overcome by adding the condition $(d\tilde{\mathbf{u}}_L/d\sigma)(\lambda_k(\mathbf{u}_L)) = \mathbf{0}$. However, from the existence and uniqueness results established by Alouges and Merlet for these systems, we see that the only solution which satisfies these initial conditions for this system is the trivial solution, i.e. $\tilde{\mathbf{u}}_L \equiv \mathbf{u}_L, \forall\sigma$. Therefore, the Alouges–Merlet approximate shock curves are indeed reversible. □

### 2.4. An application to DLM path theory

Alouges and Merlet define their approximate shock curves by differentiating the Rankine–Hugoniot jump condition in the conservative case. Alternatively, we could consider the DLM Rankine–Hugoniot jump condition (2.1) for some chosen path $\phi(s;\mathbf{u}_L,\mathbf{u}_R)$, and consider the right state, $\mathbf{u}_R$ as a function of the shock speed $\sigma$,

$$\sigma(\mathbf{u}_R(\sigma)-\mathbf{u}_L) = \int_0^1 A(\phi(s;\mathbf{u}_L,\mathbf{u}_R(\sigma)))\phi_s(s;\mathbf{u}_L,\mathbf{u}_R(\sigma))\,ds.$$

Then, assuming $\phi(s;\mathbf{u}_L,\mathbf{u}_R(\sigma))$ is sufficiently smooth in $\sigma$, we can differentiate with respect to $\sigma$ to obtain,

$$\sigma\frac{d\mathbf{u}_R}{d\sigma} + \mathbf{u}_R - \mathbf{u}_L = \int_0^1 A(\phi)\phi_{s\sigma} + DA(\phi)(\phi_\sigma,\phi_s)\,ds.$$

Here we have use the notation $\phi=\phi(s;\mathbf{u}_L,\mathbf{u}_R(\sigma))$ and $DA(\mathbf{u})(\mathbf{v},\mathbf{w}) = (\sum_{i=1}^n(\partial A(\mathbf{u})/\partial u_i)v_i)\cdot\mathbf{w}$. Integrating the first term in the integrand by parts we obtain

$$\sigma\frac{d\mathbf{u}_R}{d\sigma} + \mathbf{u}_R - \mathbf{u}_L = [A(\phi)\phi_\sigma|_{s=0}^{s=1} + \int_0^1 DA(\phi)(\phi_\sigma,\phi_s) - DA(\phi)(\phi_s,\phi_\sigma)\,ds,$$

and using the fact that $\phi(0;\mathbf{u}_L,\mathbf{u}_R(\sigma))=\mathbf{u}_L$ and $\phi(1;\mathbf{u}_L,\mathbf{u}_R(\sigma))=\mathbf{u}_R(\sigma)$ we obtain

$$\sigma\frac{d\mathbf{u}_R}{d\sigma} + \mathbf{u}_R - \mathbf{u}_L = A(\mathbf{u}_R)\frac{d\mathbf{u}_R}{d\sigma} + \int_0^1 DA(\phi)(\phi_\sigma,\phi_s) - DA(\phi)(\phi_s,\phi_\sigma)\,ds.$$

If we denote,

$$R(\phi;\mathbf{u}_L,\mathbf{u}_R(\sigma)) = \int_0^1 DA(\phi)(D_{\mathbf{u}_R}\phi,\phi_s) - DA(\phi)(\phi_s,D_{\mathbf{u}_R}\phi)\,ds, \tag{2.14}$$

then we obtain the following differential system which describes the DLM shock curves:

$$\begin{cases} (\tilde{A}(\mathbf{u}_R)-\sigma I)\dfrac{d\mathbf{u}_R}{d\sigma} = \mathbf{u}_R-\mathbf{u}_L, \\ \mathbf{u}_R(\lambda_k(\mathbf{u}_L)) = \mathbf{u}_L. \end{cases} \tag{2.15}$$

Here $\tilde{A}(\mathbf{u}_R) = A(\mathbf{u}_R) - R(\phi;\mathbf{u}_L,\mathbf{u}_R(\sigma))$. Notice that $R(\phi;\mathbf{u}_L,\mathbf{u}_R(\sigma))=O(|\sigma-\lambda_k(\mathbf{u}_L)|)$, and hence $\tilde{A}(\mathbf{u}_L)=A(\mathbf{u}_L)$, and in a sufficiently small neighborhood of $\mathbf{u}_L$, $\tilde{A}(\mathbf{u}_R)$ will have $n$ distinct real eigenvalues. Let us denote the eigenvalues and eigenvectors of $\tilde{A}(\mathbf{u})$ by $\tilde{\lambda}_1(\mathbf{u}),\ldots,\tilde{\lambda}_n(\mathbf{u})$ and $\tilde{\mathbf{r}}_1(\mathbf{u}),\ldots,\tilde{\mathbf{r}}_n(\mathbf{u})$, respectively. Note that $\tilde{\lambda}_i(\mathbf{u}_L)=\lambda_i(\mathbf{u}_L)$ and $\tilde{\mathbf{r}}_i(\mathbf{u}_L)=\mathbf{r}_i(\mathbf{u}_L)$ for all $i$. Again this system therefore suffers from the same degeneracy as the original systems studied by Alouges and Merlet. Fortunately, the arguments made in [3] to prove Proposition 2.1, and the arguments in the proof of Proposition 2.2, can be repeated for this system and we can establish the following result.

**Proposition 2.3.** *Suppose that the k-th field is genuinely nonlinear. Then Eq. (2.15) has a unique, non-trivial solution in the neighborhood of $\lambda_k(\mathbf{u}_L)$. Moreover, the non-trivial solution satisfies*

$$\mathbf{u}_R(\sigma) = \mathbf{u}_L + \frac{2(\sigma-\lambda_k(\mathbf{u}_L))}{\nabla\tilde{\lambda}_k(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L)}\mathbf{r}_k(\mathbf{u}_L) + \frac{4(\sigma-\lambda_k(\mathbf{u}_L))^2}{(\nabla\tilde{\lambda}_k(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L))^2}D\tilde{\mathbf{r}}_k(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L) + O(|\sigma-\lambda_k(\mathbf{u}_L)|^3). \tag{2.16}$$

The proof of this proposition is analogous to the proof of Proposition 2.2, where $A(\mathbf{u})$ is replaced by $\tilde{A}(\mathbf{u})$ and we then use the fact that $\tilde{A}(\mathbf{u}_L)=A(\mathbf{u}_L)$. Note, however, that since $\tilde{\lambda}_k(\mathbf{u})$ is not necessarily equal to $\lambda_k(\mathbf{u})$, we are not guaranteed that $\nabla\tilde{\lambda}_k(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L)=1$. Hence, these terms appear in the local expansions of the DLM shock curves.

Let us proceed to investigate whether the DLM shock curves are reversible for a general path $\phi(s)$. We can easily see that the arguments made in proving Theorem 2.1 can be repeated almost entirely upon replacing $A(\mathbf{u})$ by $\tilde{A}(\mathbf{u})$. However, the problem comes when we try to invoke Lemma 2.1, for this lemma will not hold for a general path. Indeed, if we repeat the arguments in the proof of Lemma 2.1 using the decomposition in Proposition 2.3 we will find that:

$$\begin{aligned}\sigma-\lambda_k(\mathbf{u}_R) &= \sigma-\lambda_k\left(\mathbf{u}_L + \frac{2(\sigma-\lambda_k(\mathbf{u}_L))}{\nabla\tilde{\lambda}_k(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L)}\mathbf{r}_k(\mathbf{u}_L) + O(|\sigma-\lambda_k(\mathbf{u}_L)|^2)\right),\\ &= \sigma-\lambda_k(\mathbf{u}_L) - \frac{2}{\nabla\tilde{\lambda}_k(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L)}(\sigma-\lambda_k(\mathbf{u}_L)) + O(|\sigma-\lambda_k(\mathbf{u}_L)|^2)),\\ &= \left(1-\frac{2}{\nabla\tilde{\lambda}_k(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L)}\right)(\sigma-\lambda_k(\mathbf{u}_L)) + O(|\sigma-\lambda_k(\mathbf{u}_L)|^2)),\end{aligned}$$

and therefore when we find the expansion of $\tilde{\mathbf{u}}_L$ as in the proof of Theorem 2.1 we obtain

$$\tilde{\mathbf{u}}_L = \mathbf{u}_L + \frac{4}{\nabla\tilde{\lambda}_k(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L)}\left(1-\frac{1}{\nabla\tilde{\lambda}_k(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L)}\right)(\sigma-\lambda_k(\mathbf{u}_L))\mathbf{r}_k(\mathbf{u}_L) + O(|\sigma-\lambda_k(\mathbf{u}_R)|^2).$$

Hence, we see that, in general, $d\tilde{\mathbf{u}}_L/d\sigma(\lambda_k(\mathbf{u}_L)) \neq \mathbf{0}$ and therefore $\tilde{\mathbf{u}}_L(\sigma) \neq \mathbf{u}_L$ for all $\sigma$. Thus the DLM shock curves are not reversible for a general path $\phi(s)$. In fact, we can state a useful corollary.

**Corollary 2.1.** *If the DLM shock curves are reversible then they have the local expansion*

$$\mathbf{u}_R = \mathbf{u}_L + \varepsilon\mathbf{r}_k(\mathbf{u}_L) + \frac{1}{2}\varepsilon^2 D\mathbf{r}_k(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L) + O(\varepsilon^3),$$

$$\sigma = \lambda_k(\mathbf{u}_L) + \frac{1}{2}\varepsilon + O(\varepsilon^2),$$

*where $\varepsilon=\lambda_k(\mathbf{u}_R)-\lambda_k(\mathbf{u}_L)$.*

This can be equivalently stated as

**Corollary 2.2.** *The DLM shock curves are reversible if and only if $R(\phi;\mathbf{u}_L,\mathbf{u}_R(\sigma))=O(|\sigma-\lambda_k(\mathbf{u}_L)|^2)$, where $R(\phi;\mathbf{u}_L,\mathbf{u}_R(\sigma))$ is defined by (2.14).*

Therefore, when shock curves are reversible they have the same second order local expansion as the rarefaction curves near a given left state, $\mathbf{u}_L$. This is a general property of shock curves in conservative systems. Hence, reversibility gives us a simple way of specifying this property of shock curves in nonconservative systems. Moreover, from Corollaries 2.1 and 2.2, this property is easily verified by either directly looking at the local expansion of the shock curves, or by examining the remainder term $R(\phi;\mathbf{u}_L,\mathbf{u}_R(\sigma))$. While reversibility is primarily a theoretical tool, it is not unreasonable for use to desire such a property in nonconservative shock curves. Notice that if the shock curves are indeed reversible then we know that if a discontinuity wave with left state $\mathbf{u}_L$ and right state $\mathbf{u}_R$ is a weak solution of the nonconservative system, then so is the discontinuity wave with left state $\mathbf{u}_R$ and right state $\mathbf{u}_L$ (albeit one of these discontinuity waves will not be entropic). Furthermore, as we will see below in our numerical tests, shock curves which are reversible – and therefore have the local expansion (2.6) and (2.7) – show a good level of agreement with the weak solutions defined using the Alouges–Merlet shock curves.

## 3. Numerical approximation

In this section, we will focus on using the approximate shock curves as defined by Alouges and Merlet within the DLM framework in order to define and analyse several numerical schemes. In particular, we will use a Lax–Friedrichs-like scheme, based on the DLM framework, as our prototype example in our calculations, but we will also consider a Godunov scheme [15], again based in the DLM framework, in our numerical simulations. In [8], Castro et al. show that a Lax–Friedrichs-type scheme using the DLM definitions can be written as:

$$U_j^{n+1}=\frac{1}{2}(U_{j+1}^n+U_{j-1}^n)-\frac{\Delta t}{2\Delta x}(G_{j+1/2}^n+G_{j-1/2}^n). \qquad (3.1)$$

where for all $j\in\mathbb{Z}$,

$$G_{j+1/2}^n=\int_0^1 A(\psi(s;U_j^n,U_{j+1}^n))\frac{\partial\psi}{\partial s}(s;U_j^n,U_{j+1}^n)\,ds.$$

Similarly, Castro Parés et al. [6] show that a Godunov-type scheme can be written:

$$U_j^{n+1}=U_j^n-\frac{\Delta t}{\Delta x}(G_{j+1/2}^{n,-}+G_{j-1/2}^{n,+}),$$

where

$$G_{j+1/2}^{n,-}=\int_0^1 A(\psi(s;U_j^n,U_{j+1/2}^n))\frac{\partial\psi}{\partial s}(s;U_j^n,U_{j+1/2}^n)\,ds,$$

$$G_{j+1/2}^{n,+}=\int_0^1 A(\psi(s;U_{j+1/2}^n,U_{j+1}^n))\frac{\partial\psi}{\partial s}(s;U_{j+1/2}^n,U_{j+1}^n)\,ds,$$

and $U_{j+1/2}^n$ is the value at $x=0$ of the solution to the Riemann Problem

$$U(x,0)=\begin{cases}U_j^n, & x<0,\\ U_{j+1}^n, & x>0.\end{cases}$$

In both these schemes, the function $\psi(s;\mathbf{u}_L,\mathbf{u}_R)$ is a family of paths connecting $\mathbf{u}_L$ to $\mathbf{u}_R$, which we are free to choose. However, as detailed by Castro et al., a good choice should be to choose $\psi(s;\mathbf{u}_L,\mathbf{u}_R)$ as the union of the $k$-simple curves which solve the Riemann problem with left state $\mathbf{u}_L$ and right state $\mathbf{u}_R$, and re-parametrized so that $\psi(0;\mathbf{u}_L,\mathbf{u}_R)=\mathbf{u}_L$ and $\psi(1;\mathbf{u}_L,\mathbf{u}_R)=\mathbf{u}_R$. The $k$-simple curves are comprised of the $k$-rarefaction curves, the $k$-contact discontinuity curves, and the path $\phi$, which was used to defined the DLM jump condition and the shock curves. This choice of path was already proposed in [20]. We call such a path a 'Godunov path' for brevity and give a formal definition:

**Definition 3.1** *(Godunov Path)*. Suppose we have selected a family of shock curves for a nonconservative system, defined by (2.1), with some chosen family of paths $\phi$. Consider the Riemann problem with left state $\mathbf{u}_L$ and right state $\mathbf{u}_R$ and suppose the unique entropy solution of this Riemann problem exists and consists of the $n+1$ constant states $\mathbf{u}_L=\mathbf{u}_0,\mathbf{u}_1,\ldots,\mathbf{u}_{n-1},\mathbf{u}_n=\mathbf{u}_R$ where $\mathbf{u}_0$ is connected to $\mathbf{u}_1$ by a 1-simple wave, $\mathbf{u}_1$ is connected to $\mathbf{u}_2$ by a 2-simple wave, and so on. The *Godunov path*, $\psi(s;\mathbf{u}_L,\mathbf{u}_R)$, of this Riemann problem, with respect to the chosen shock curves, is a parametrization (at least $C^2$) of the rarefaction curves, contact discontinuity curves, and the path $\phi$, linking $\mathbf{u}_L$ to $\mathbf{u}_R$, such that

$$\psi(0;\mathbf{u}_L,\mathbf{u}_R)=\mathbf{u}_L,\qquad \psi(1;\mathbf{u}_L,\mathbf{u}_R)=\mathbf{u}_R.$$

**Remark 3.1.** From the existence and uniqueness theorems regarding shock, rarefaction, and contact discontinuity curves we know that for a fixed $\mathbf{u}_L$ there will exist a neighborhood $\mathcal{N}$ of $\mathbf{u}_L$ which for all $\mathbf{u}_R\in\mathcal{N}$, the Godunov path $\psi(s;\mathbf{u}_L,\mathbf{u}_R)$ will exist and be at least piecewise $C^2$.

**Remark 3.2.** From the definition of the Alouges–Merlet shock curves, we see that they can be view as DLM shock curves where the path $\phi$ is the Alouges–Merlet shock curve itself. Therefore, the Godunov path in this case is simply the union of the $k$-simple curves (including the $k$-shock curves) which link $\mathbf{u}_L$ and $\mathbf{u}_R$.

Although the computation of the path $\psi$ is clearly very computationally complex, at this point our goal is not to propose a fast numerical scheme for approximating NCHSs, but rather to propose several numerical schemes which do not suffer from such severe convergence errors. We will, however, suggest some ways in which using approximate Riemann solvers (i.e. an approximations of the path $\psi$) may help to create faster schemes in Section 4.

### 3.1. Towards a Lax–Wendroff-type theorem

We are interested in this paper in the question of what weak solutions the numerical approximations of nonconservative systems converge to that end, let us briefly discuss how the well-know Lax–Wendroff theorem fails for nonconservative systems and let us outline how knowledge of the Alouges–Merlet approximate shock curves may help us construct schemes which converge closely to the their approximate vanishing viscosity solutions. In what follows, we do not focus on making a fully rigorous argument and presenting a Lax–Wendroff-type theorem. Rather, we aim to motivate how the Alouges–Merlet shock curves may help to reduce the appearance of convergence errors in numerical schemes.

To begin, assume that $A$ is a smooth matrix-valued function, and let $\psi$ be the Godunov path, defined using the DLM jump condition relative to a chosen family of paths $\phi$. For $\Delta x>0$, let $U_{\Delta x}$ be the numerical solution obtained from the Lax–Friedrichs-type scheme

$$U_j^{n+1}=\frac{1}{2}(U_{j+1}^n+U_{j-1}^n)-\frac{\Delta t}{2\Delta x}(G_{j+}^n1/2+G_{j-1/2}^n)$$

and let $U_{\Delta x}(x,0)$ be the initial data. Furthermore, assume that

$$\|U_h\|_{L^\infty(\mathbb{R}\times\mathbb{R}_+)}\leqslant C$$

and that there exists a function $U\in(L^\infty(\mathbb{R}\times\mathbb{R}_+)\cap BV(\mathbb{R}\times\mathbb{R}_+))^m$ such that $U_{\Delta x}\to U$ in $L^1_{loc}(\mathbb{R}\times\mathbb{R}_+)$ as $\Delta x\to 0$. Our question is: what is $U$? More importantly, if we cannot determine $U$ exactly, how can we approximate it?

Let us denote by $\nu$ a test function of $C^1(\mathbb{R}\times\mathbb{R}_+)$. Denoting $\nu_j^n=\nu(x,t)$ for some $(x,t)\in(x_{j-1/2},x_{j+1/2})\times[t_n,t_{n+1})$, we multiply (3.1) by

$v_j^n$ sum over $j$ and $n$ to obtain:

$$\sum_{j=-\infty}^{+\infty}\sum_{n=0}^{+\infty}(U_j^{n+1}-U_j^n)v_j^n+\frac{\Delta t}{2\Delta x}(G_{j+1/2}^n+G_{j-1/2}^n)v_j^n-(1/2)\times(U_{j+1}^n-U_j^n)v_j^n+(1/2)(U_j^n-U_{j-1}^n)v_j^n=\mathbf{0}.$$

Using summation by parts on each of the terms, and multiplying by $\Delta x$, we obtain:

$$\sum_{j=-\infty}^{+\infty}\sum_{n=0}^{+\infty}\left[U_j^n\frac{v_j^{n+1}-v_j^n}{\Delta t}+\left(\frac{1}{2}\sum_{k=-\infty}^{j}G_{k+1/2}^n+G_{k-1/2}^n\right)\frac{v_{j+1}^n-v_j^n}{\Delta x}\right]\Delta t\Delta x+\sum_{j=-\infty}^{+\infty}U_j^0v_j^0\Delta x=\mathbf{0} \tag{3.2}$$

The only difference between (3.2) and the usual expression in the proof of the Lax–Wendroff theorem is this term $(1/2)\sum_{k=-\infty}^{j}G_{k+1/2}^n+G_{k-1/2}^n$. For a conservative scheme, when $G_{k+1/2}^n$ reduces to $\mathbf{f}(U_{j+1}^n)-\mathbf{f}(U_j^n)$, this sum would telescope to $1/2(\mathbf{f}(U_{j+1}^n)+\mathbf{f}(U_j^n))$, which approaches the flux function $\mathbf{f}(\mathbf{u})$ as $\Delta x\to 0$. In this nonconservative case, we want that,

$$\frac{1}{2}\sum_{k=-\infty}^{j}G_{k+1/2}^n+G_{k-1/2}^n\to\int_{-\infty}^{x}[A(U)U_x]_\phi\cdot\mathbf{1}\,dx, \tag{3.3}$$

when $\Delta x\to 0$ in such a way that $x_j\to x$, and convergence is in the sense of measures. From the definition of the measure $[A(U)U_x]_\phi$ we must consider two cases: regions where $U$ is $C^1$ and regions where $U$ is discontinuous. To begin, suppose at $U$ is differentiable at $x$. Then, from the definition of the Godunov path $\psi$ it is easy to verify that

$$G_{j+1/2}^n=\int_0^1 A(\psi(s;U_j^n,U_{j+1}^n))\frac{\partial\psi}{\partial s}(s;U_j^n,U_{j+1}^n)\,ds\to A(U)U_x,$$

as $\Delta x\to 0$. On the other hand, suppose $U$ is discontinuous at $x$ with left state $U_L$ and right state $U_R$. In this case, for every $\Delta x>0$ there is some segment of $U_j^n$ which is 'close' to the viscous profile of this shock wave, relative to the numerical viscosity of the scheme (plus higher order viscous terms). Let us define indices $l$ and $j$ (which depend on $\Delta x$) such that $x_l$, $x_j\to x$, and $U_l^n$ and $U_j^n$ are the left and right 'ends' of this viscous profile, i.e. $U_l^n\to U_L$, and $U_j^n\to U_R$ as $\Delta x\to 0$. In this case, the condition (3.3) reads,

$$\frac{1}{2}\sum_{k=l}^{j-1}G_{k+1/2}^n+G_{k-1/2}^n\to\int_0^1 A(\phi(s;U_L,U_R))\frac{\partial\phi}{\partial s}(s;U_L,U_R)\,ds,$$

or, inserting the definition of $G_{j+1/2}^n$,

$$\sum_{k=l}^{j-1}\int_0^1 A(\psi(s;U_k^n,U_{k+1}^n))\frac{\partial\psi}{\partial s}(s;U_k^n,U_{k+1}^n)\,ds+\int_0^1 A(\psi(s;U_{k-1}^n,U_k^n))\frac{\partial\psi}{\partial s}(s;U_{k-1}^n,U_k^n)\,ds\to\int_0^1 A(\phi(s;U_L,U_R))\frac{\partial\phi}{\partial s}(s;U_L,U_R)\,ds. \tag{3.4}$$

Hence, if (3.4) holds then the sum of the integrals of the Godunov paths in $G_{k+1/2}^n$ converge to the proper nonconservative product and we would expect that the numerical scheme would converge to the chosen weak solution. This is clearly not true for a general choice of path, however. This is because the viscous profile through which we sum $G_{k+1/2}^n$ will depend on the numerical viscosity of the scheme, and furthermore $\phi$ itself may be very different from this viscous profile. Note, however, that if we choose the Alouges–Merlet shock curves as our path $\phi$, and use a numerical scheme whose numerical viscosity commutes with $A(\mathbf{u})$, then the viscous profile itself will be close to the path in the nonconservative product, and we would expect the convergence of the numerical scheme to be much better. In the section below, we will investigate when/how we may approximate the limit of the sum in (3.4).

### 3.2. Godunov path with reversible shock curves

In this section we will show that through a modification to the definition of the Godunov path $\psi$ using reversible shock curves, we are able to approximate the limit of the sum in (3.4). As defined above, the Godunov path is found by solving the Riemann problem and linking the intermediate states using the rarefaction curves, contact discontinuity curves, and the path $\phi$. Let us instead construct a *modified* Godunov path by linking the intermediate states using the shock curves, rather than the path itself. We can then establish the following property about integrating through reversible shock curves.

**Proposition 3.1.** *Suppose we have chosen a set of reversible shock curves. Let $\mathbf{u}_R$ be a state on the k-shock curve, $\mathcal{S}_k(\mathbf{u}_L)$, of the left state $\mathbf{u}_L$. Let $\mathbf{u}_1$ be a another state on this shock curve between $\mathbf{u}_L$ and $\mathbf{u}_R$. Furthermore, let $\psi$ be the modified Godunov path defined using these reversible shock curves. Then*

$$\int_0^1 A(\psi(s;\mathbf{u}_L,\mathbf{u}_1))\frac{\partial\psi}{\partial s}(s;\mathbf{u}_L,\mathbf{u}_1)\,ds+\int_0^1 A(\psi(s;\mathbf{u}_1,\mathbf{u}_R))\frac{\partial\psi}{\partial s}(s;\mathbf{u}_1,\mathbf{u}_R)\,ds=\int_0^1 A(\psi(s;\mathbf{u}_L,\mathbf{u}_R))\frac{\partial\psi}{\partial s}(s;\mathbf{u}_L,\mathbf{u}_R)\,ds+O(||\mathbf{u}_R-\mathbf{u}_L||^3). \tag{3.5}$$

**Proof.** Since the shock curves chosen to define the Godunov path are reversible, from Proposition 2.2 the $k$-shock curve linking $\mathbf{u}_L$, $\mathbf{u}_1$ and $\mathbf{u}_R$ will have the local expansion (2.6). Thus, since $\psi$ is a reparametrization of this shock curve, we will have the following expansions:

$$\psi(s;\mathbf{u}_L,\mathbf{u}_1)=\mathbf{u}_L+\varepsilon_1 s\mathbf{r}_k(\mathbf{u}_L)+\frac{1}{2}\varepsilon_1^2s^2D\mathbf{r}_k(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L)+O(\varepsilon_1^3), \tag{3.6}$$

$$\psi(s;\mathbf{u}_1,\mathbf{u}_R)=\mathbf{u}_1+\varepsilon_2 s\mathbf{r}_k(\mathbf{u}_1)+\frac{1}{2}\varepsilon_2^2s^2D\mathbf{r}_k(\mathbf{u}_1)\cdot\mathbf{r}_k(\mathbf{u}_1)+O(\varepsilon_2^3), \tag{3.7}$$

$$\psi(s;\mathbf{u}_L,\mathbf{u}_R)=\mathbf{u}_L+\varepsilon s\mathbf{r}_k(\mathbf{u}_L)+\frac{1}{2}\varepsilon^2s^2D\mathbf{r}_k(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L)+O(\varepsilon^3), \tag{3.8}$$

where $\varepsilon_1=\lambda_k(\mathbf{u}_1)-\lambda_k(\mathbf{u}_L)$, $\varepsilon_2=\lambda_k(\mathbf{u}_R)-\lambda_k(\mathbf{u}_1)$, and $\varepsilon=\lambda_k(\mathbf{u}_R)-\lambda_k(\mathbf{u}_L)=\varepsilon_1+\varepsilon_2$. Evaluating (3.6) at $s=1$, we obtain the expressions,

$$\mathbf{u}_1=\mathbf{u}_L+\varepsilon_1\mathbf{r}_k(\mathbf{u}_L)+\frac{1}{2}\varepsilon_1^2D\mathbf{r}_k(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L)+O(\varepsilon_1^3). \tag{3.9}$$

Using this in (3.7), we obtain

$$\psi(s;\mathbf{u}_1,\mathbf{u}_R)=\mathbf{u}_L+(\varepsilon_1+\varepsilon_2 s)\mathbf{r}_k(\mathbf{u}_L)+\frac{1}{2}(\varepsilon_1^2+2\varepsilon_1\varepsilon_2 s+\varepsilon_2^2s^2)D\mathbf{r}_k(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L)+O(\varepsilon_1^3)+O(\varepsilon_2^3), \tag{3.10}$$

Finally, inserting (3.6) and (3.10) into the left hand side of (3.5) we obtain after a long calculation

$$(\varepsilon_1+\varepsilon_2)A(\mathbf{u}_L)\mathbf{r}_k(\mathbf{u}_L)+\frac{1}{2}(\varepsilon_1^2+2\varepsilon_1\varepsilon_2+\varepsilon_2^2)(A(\mathbf{u}_L)D\mathbf{r}_k(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L)$$
$$+DA(\mathbf{u}_L)(\mathbf{r}_k(\mathbf{u}_L),\mathbf{r}_k(\mathbf{u}_L)))+O(\varepsilon_1^3)+O(\varepsilon_2^3),$$

and inserting (3.8) into the right hand side of (3.5) we obtain

$$\varepsilon A(\mathbf{u}_L)\mathbf{r}_k(\mathbf{u}_L)+\frac{1}{2}\varepsilon^2(A(\mathbf{u}_L)D\mathbf{r}_k(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L)$$
$$+DA(\mathbf{u}_L)(\mathbf{r}_k(\mathbf{u}_L),\mathbf{r}_k(\mathbf{u}_L)))+O(\varepsilon^3).$$

Recalling that $\varepsilon=\varepsilon_1+\varepsilon_2$ proves the result. □

**Remark 3.3.** There is a connection here between Proposition 3.1 and the work done by Castro et al. in [7]. In this paper, the authors calculate the equivalent equation for the Lax–Friedrichs-like scheme in order to study its higher order viscous terms. The equivalent equation is calculated by assuming that the solution is smooth at a certain point and performing a Taylor expansion of the numerical scheme. In this calculation, the authors show that the equivalent equation of scheme (3.1) is

$$\mathbf{u}_t+A(\mathbf{u})\mathbf{u}_x=\frac{\Delta x^2}{2\Delta t}\mathbf{u}_{xx}-\frac{\Delta t}{2}\mathbf{u}_{tt}+\frac{\Delta x}{2}I_2(\mathbf{u})+O(\Delta x^2)+O(\Delta t^2),$$

where

$$I_2(\mathbf{u})=\int_0^1 DA(\mathbf{u})(D_{\mathbf{u}_L}\psi\cdot\mathbf{u}_x,D_{\mathbf{u}_L}\psi_s\cdot\mathbf{u}_x)\,ds$$
$$+\int_0^1 DA(\mathbf{u})(D_{\mathbf{u}_R}\psi\cdot\mathbf{u}_x,D_{\mathbf{u}_R}\psi_s\cdot\mathbf{u}_x)\,ds.$$

Interestingly, if we likewise assume that the solution is smooth and perform a Taylor series expansion of (3.5) around $\mathbf{u}=\mathbf{u}_1$ we obtain after a lengthy calculation,

$$\Delta x^2 I_2(\mathbf{u})=O(\Delta x^3).$$

Therefore, in the context of the equivalent equation of the numerical scheme, Proposition 3.1 tells us that when the numerical solution lies along a shock curve, $I_2(\mathbf{u})=O(\Delta x)$.

Let us return to the sum in (3.4),

$$\sum_{k=l}^{j-1}G_{k+1/2}^n=\sum_{k=l}^{j}\int_0^1 A(\psi(s;U_k^n,U_{k+1}^n))\frac{\partial\psi}{\partial s}(s;U_k^n,U_{k+1}^n)\,ds.$$

If we assume that the numerical scheme we are using adds a numerical viscosity $B(\mathbf{u})$ which commutes with $A(\mathbf{u})$ then the viscous profile through which this sum will be close to the Alouges–Merlet shock curves to $O(||\mathbf{u}_R-\mathbf{u}_L||^3)$ and $O(\Delta x^2)$ (due to higher order viscous terms). Furthermore, Proposition 3.1 tells us that along a reversible shock curve this sum will telescope to the third order in the strength of the shock. Hence we obtain, at least formally (as uniform convergence of the solution in the sense of graph may be needed)

$$\sum_{k=l}^{j-1}G_{k+1/2}^n=\int_0^1 A(\psi(s;U_l^n,U_j^n))\frac{\partial\psi}{\partial s}(s;U_l^n,U_j^n)\,ds$$
$$+O(||U_j^n-U_l^n||^3)+O(\Delta x^2),$$
$$\to\int_0^1 A(\psi(s;U_L,U_R))\frac{\partial\psi}{\partial s}(s;U_L,U_R)\,ds+O(||U_R-U_L||^3)$$

as $\Delta x\to 0$. We can therefore expect that the limit of the left hand side of (3.4) will be close to the proper nonconservative product, to the third order. Since, this not a rigorous argument we state this formally as a conjecture and defer a formal proof to a future work.

**Conjecture 3.1.** *Assume that we have chosen some set of reversible shock curves, and suppose we are approximating a nonconservative system with a Parés-like scheme based on the DLM path-theory. Suppose the numerical path $\psi$ in this scheme is a modified Godunov path, as described above, defined using the reversible shock curves. Furthermore, suppose the numerical viscosity of the scheme, $B(\mathbf{u})$, is of the form,*

$$\mathbf{u}_t+A(\mathbf{u})\mathbf{u}_x=\varepsilon(B(\mathbf{u})\mathbf{u}_x)_x$$

*such that $[A(\mathbf{u}),B(\mathbf{u})]=0$. Then if the numerical solution $U_j^n$ is converging to a function $U\in(L^\infty(\mathbb{R}\times\mathbb{R}_+)\cap BV(\mathbb{R}\times\mathbb{R}_+))^m$, then $U$ is a weak solution of the nonconservative system and it will agree with the Alouges–Merlet viscous solutions near a discontinuity, to third order in the strength of the discontinuity.*

## 4. Scheme acceleration

In this section we will briefly discuss possible techniques for reducing the computational complexity of the schemes presented above and some alternative numerical schemes which are significantly less computationally costly, but whose numerical solutions still exhibit good agreement with the Alouges–Merlet vanishing viscosity solutions. We present the numerical results or these schemes, along with the results from the numerical schemes which use the fully computed Godunov path, in Section 5.

Clearly, the main computational expense of these numerical scheme is focused in the computation of the Godunov path, $\psi$. Note as well that for schemes which use exact Riemann solvers (i.e. Godunov schemes) we must still solve each interface Riemann problem to determine the interface state $U_{j+(1/2)}$. Hence, even if a very simple numerical path $\psi$ is used, the determination of this interface state is still expensive. Therefore, when using a approximation of the Godunov path we also use a numerical scheme which uses an approximate Riemann solver (i.e. the Lax–Friedrichs-like scheme, or Roe scheme) to avoid this expensive calculation. We propose the following two strategies:

- Approximate the shock/rarefaction/contact discontinuity curves in the construction of the Godunov path, $\psi$.
- Approximate the whole Godunov path $\psi$ by a simpler path $\tilde{\psi}$, which still has the local approximation (2.6) along shock profiles.

Let us examine these strategies in more detail.

### 4.1. Approximate the shock/rarefaction/contact discontinuity curves

From the computational complexity point of view, the main issue of the above schemes is the numerical integration of the shock, rarefaction, and contact discontinuity curves, and the determination of intermediate states. However, a simple alternative consists of using the local parametrization of each of these simple curves. Let us demonstrate this through a simple example by considering a $2\times 2$ system, and let us assume that we are approximating the modified Godunov path, using reversible shock curves, for the Riemann problem with left state $\mathbf{u}_L$ and right state $\mathbf{u}_R$. This amounts to finding the intersection of the 1- and 2-simple curves which connect $\mathbf{u}_L$ and $\mathbf{u}_R$. From Corollary 2.1, we know all the reversible shock curves, rarefaction curves, and contact discontinuity curves have the local approximations:

$$\mathcal{S}_k(\mathbf{u})\sim\mathbf{u}+\varepsilon\mathbf{r}_k(\mathbf{u})+\frac{1}{2}\varepsilon^2 D\mathbf{r}_k(\mathbf{u})\cdot\mathbf{r}_k(\mathbf{u}).$$

Therefore, the problem of approximating the modified Godunov path for this $2 \times 2$ system reduces to finding $\varepsilon_1$ and $\varepsilon_2$ such that,

$$\mathbf{u}_1 = \mathbf{u}_L + \varepsilon_1 \mathbf{r}_1(\mathbf{u}_L) + \frac{1}{2}\varepsilon_1^2 D\mathbf{r}_1(\mathbf{u}_L)\cdot \mathbf{r}_1(\mathbf{u}_L),$$
$$\mathbf{u}_R = \mathbf{u}_1 + \varepsilon_2 \mathbf{r}_2(\mathbf{u}_1) + \frac{1}{2}\varepsilon_2^2 D\mathbf{r}_2(\mathbf{u}_1)\cdot \mathbf{r}_2(\mathbf{u}_1),$$

which can be solved quickly using root-finding software, for sufficiently small jumps. Clearly, this process induces an error that is of third order in the strength of the jump and is naturally not accurate for strong discontinuities. However, one can implement this approximate modified Godunov path when $||U_{j+1}^n - U_j^n||$ is smaller than a certain given tolerance. In this way we use the approximate simple curves to accurately approximate the modified Godunov path at this interface, and thereby avoid solving the differential systems, while preserving the accurate Godunov path at other interfaces where the jumps are large.

### 4.2. Approximate the whole Godunov path $\psi$ by a simpler path $\tilde{\psi}$

Another approach to the design of a fast numerical scheme would be to replace the entire Godunov path $\psi$ by some approximation $\tilde{\psi}$, which we carefully choose so that it still has the local expansion (2.6) along shock profiles. While this could be done in many ways let us consider a particularly simple approximation: a straight line. Thus, we approximate the entire path $\psi(s\,;\mathbf{u}_L, \mathbf{u}_R)$ by the straight line:

$$\tilde{\psi}(s;\mathbf{u}_L,\mathbf{u}_R) = \mathbf{u}_L + s(\mathbf{u}_R - \mathbf{u}_L). \tag{4.1}$$

Let us consider this simple choice of path in a Roe scheme. This scheme was initially considered by Toumi [28], in which the author proposes a generalized definition of Roe's linearization for nonconservative systems based on DLM path-theory. A function $A:\mathbb{R}^m\times\mathbb{R}^m \to M_m(\mathbb{R})$ is called a (generalized) Roe linearization if

1. For all $\mathbf{v}, \mathbf{u} \in \Omega$,

$$A(\mathbf{v},\mathbf{u})(\mathbf{v}-\mathbf{u}) = \int_0^1 A(\psi(s;\mathbf{u},\mathbf{v}))\frac{\partial \psi}{\partial s}(s;\mathbf{u},\mathbf{v})\,ds.$$

2. For all $\mathbf{v}, \mathbf{u} \in \Omega$, A($\mathbf{v}$,$\mathbf{u}$) has $m$ independent eigenvectors.
3. For all $\mathbf{v} \in \Omega$,

$$A(\mathbf{v},\mathbf{v}) = A(\mathbf{v}).$$

Choosing the straight line in place of $\psi$ yields a Roe matrix

$$A_{j+1/2}^n = A(U_{j+1}^n, U_j^n) = \int_0^1 A(U_j^n + s(U_{j+1}^n - U_j^n))\,ds,$$

and the Roe scheme can be written:

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}(A_{j-1/2}^{n,+}(U_j^n - U_{j-1}^n) + A_{j+1/2}^{n,-}(U_{j+1}^n - U_j^n)), \tag{4.2}$$

where

$$A_{j+1/2}^{n,\pm} = R_{j+1/2}^n D_{j+1/2}^{n,\pm} {R_{j+1/2}^n}^{-1}.$$

and $D_{j+1/2}^n = \mathrm{diag}(\lambda_{1,j+1/2}^n, \ldots, \lambda_{m,j+1/2}^n)$ is a diagonal matrix of eigenvalues of $A_{j+1/2}^n$ and $R_{j+1/2}^n$ is the matrix of right eigenvectors of $A_{j+1/2}^n$.

While this is clearly a very simple and easy way to create a fast numerical scheme for approximating a nonconservative system, it is clear that we have sacrificed a great deal of accuracy by choosing a straight line to approximate the Godunov path. However, a direct calculation shows that for this straight path $\tilde{\psi}$ Proposition 3.1 will still hold. Therefore, we still expect the numerical solutions of this scheme to agree with the Alouges–Merlet vanishing viscosity solutions, to third order in the strength of the discontinuities.

Another simple choice for this approximate path $\tilde{\psi}$, which has the local approximation (2.6) along shock profiles, is

$$\tilde{\psi}(s;\mathbf{u}_L,\mathbf{u}_R) = \mathbf{u}_L + \left(\sum_{k=1}^m \varepsilon_k \mathbf{r}_k(\mathbf{u}_L)\right)s + \frac{1}{2}\left(\sum_{k=1}^m \varepsilon_k^2 D\mathbf{r}_k(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L)\right)s^2, \tag{4.3}$$

where the $\varepsilon_k$ are found specifically so that $\tilde{\psi}(1;\mathbf{u}_L,\mathbf{u}_R) = \mathbf{u}_R$. This choice is potentially more accurate than the straight path approximation.

## 5. Numerical results

In this section we will examine how well the numerical solutions from several schemes compare with the Alouges–Merlet vanishing viscosity solutions. We apply these schemes to a nonconservative system considered by Castro et al. in [7],

$$\begin{cases} h_t + q_x = 0, \\ q_t + \left(\dfrac{q^2}{h}\right)_x + qhh_x = 0. \end{cases} \tag{5.1}$$

This system has the form $\mathbf{v}_t + A(\mathbf{v})\mathbf{v}_x = 0$ where

$$v = \begin{pmatrix} h \\ q \end{pmatrix}, \qquad A(\mathbf{v}) = \begin{pmatrix} 0 & 1 \\ -u^2 + uh^2 & 2u \end{pmatrix},$$

and $u = q/h$. When $0 < q$ and $0 < h < (16q)^{1/3}$, this system is strictly hyperbolic and its characteristic fields are genuinely nonlinear. We choose this system for is simplicity - since the eigenvalues and eigenvectors of $A(\mathbf{v})$ can be explicitly calculated – and also to establish a comparison with the numerical benchmarks presented by Castro et al. in [7]. We consider three schemes in particular: the Lax–Friedrichs-like scheme (3.1) with the Godunov path constructed using the Alouges–Merlet shock curves, and two schemes which use the approximate paths $\tilde{\psi}$ discussed in Section 4.2. For each scheme, we calculate the numerical 1-shock curve and compare it with the exact Alouges–Merlet 1-shock curve in order to observe how well the numerical scheme approximates the Alouges–Merlet vanishing viscosity solutions.

### 5.1. Lax–Friedrichs with Alouges–Merlet shock curves

Our first test is approximating the system (5.1) using the Lax–Friedrichs-like scheme (3.1) where the numerical path $\psi$ is chosen to be the Godunov path (see Definition 3.1), defined using the Alouges–Merlet shock curves as the path, $\phi$.

In Fig. 3, we show the results of this scheme when applied to the Riemann problem,

$$\mathbf{v}(x,0) = \begin{cases} \begin{pmatrix} 1 \\ 1 \end{pmatrix} & x < 0, \\ \begin{pmatrix} 1.8 \\ 0.3 \end{pmatrix} & x > 0. \end{cases}$$

We show the results of this computation at $t = 0.5$ on a uniform mesh of $N = 400$ elements. We also show the exact Alouges–Merlet weak solution which is calculated using the DLM framework with the Alouges–Merlet shock curves as the chosen path, $\phi$. We see from this figure that the numerical solution agrees quite closely with the theoretically specified weak solution. To more completely test this

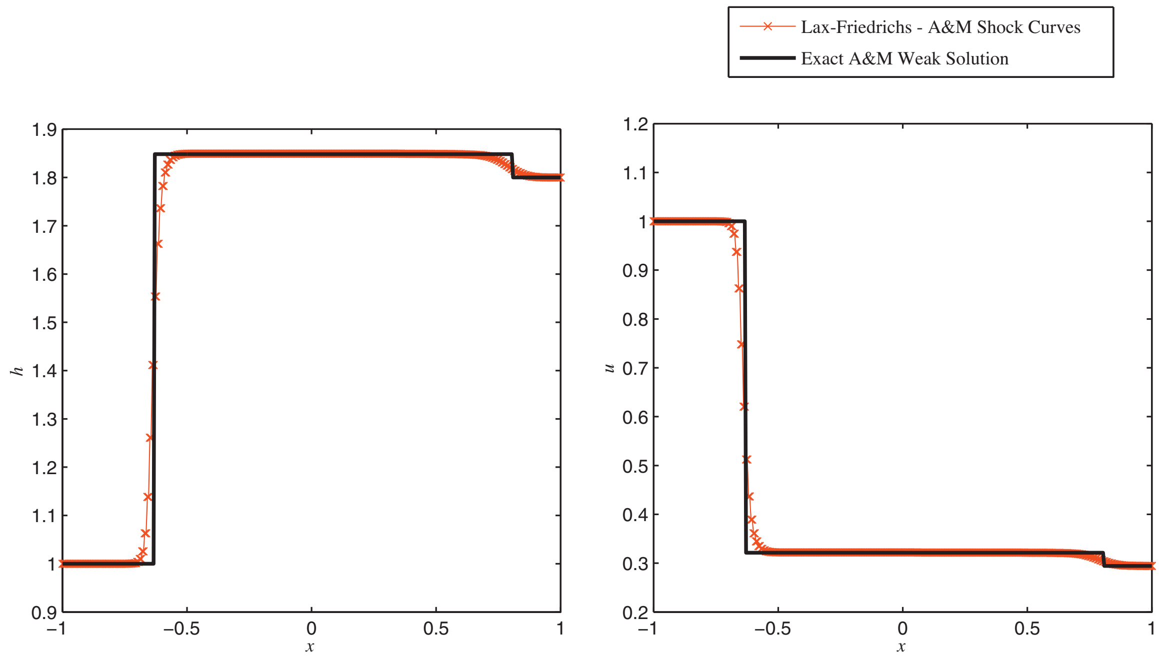


**Fig. 3.** Lax–Friedrichs: comparison exact and approximate solutions with a A&M path.

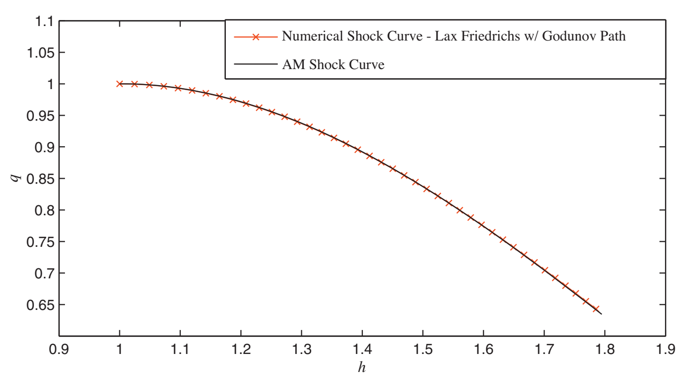


**Fig. 4.** Godunov scheme with A&M path.

agreement, we repeat this test for many Riemann problems while holding the left state $\mathbf{v}_L = (1, 1)^T$ constant and recording the intermediate state connected to $\mathbf{v}_L$ by a 1-shock wave. We show the results of this test in Fig. 4 in which we see that the numerical 1-shock curve resolved by this numerical scheme agrees quite closely with the Alouges–Merlet 1-shock curve. We also note that this close agreement of the numerical solution and the Alouges–Merlet weak solution has been observed for several other schemes using the full Godunov path, e.g. Godunov scheme, Lax–Wendroff scheme, VFFC scheme [14], and others.

As mentioned above, calculating the exact Godunov path using the Alouges–Merlet shock curves for this $2 \times 2$ system requires that at every cell interface we solve four ODE systems and find the unique intersection of the resulting simple curves. Although this is a very computationally expensive procedure, our primary goal is the construction of schemes which exhibit small convergence errors to the vanishing viscosity solutions. We consider schemes which are less computationally expensive in our next test.

### 5.2. *Numerical schemes using approximate paths $\tilde{\psi}$*

In our second numerical test, we consider two numerical schemes which use the approximate paths $\tilde{\psi}$ as described in

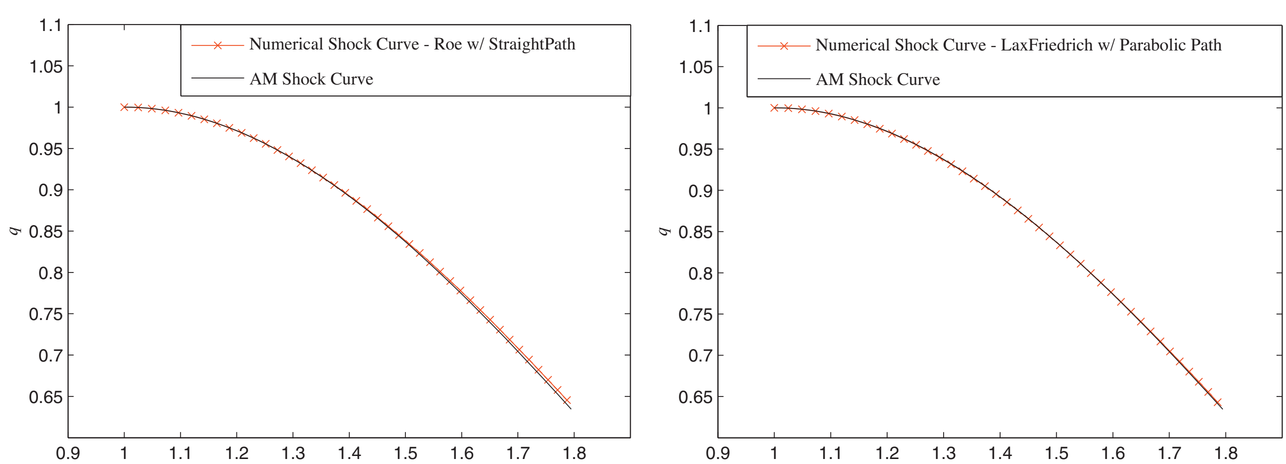


**Fig. 5.** Roe scheme with straight path (left) and Lax–Friedrichs scheme with parabolic path (right).

Section 4.2. These paths are much simpler to construct than the full Godunov path, and are therefore far less computationally expensive. However, it is unclear a priori how these approximations of the full Godunov path will affect the numerical solution of the scheme and, more specifically, to what degree the numerical shock curves of the schemes will agree with the Alouges–Merlet shock curves.

The first scheme we consider is a Roe scheme (4.2) using the straight path approximation (4.1). The numerical 1-shock curve of this scheme was calculated as it was above for the Lax–Friedrichs-like scheme and shown on the left of Fig. 5. We see in this figure that the numerical 1-shock curve of the Roe scheme with this approximate path still exhibits a good level of agreement with the Alouges–Merlet shock curve, although a small error is visible for stronger shock waves. The second scheme is again the Lax–Friedrichs-like scheme, this time using the parabolic approximate path $\tilde{\psi}$ in (4.3). The numerical 1-shock curve of this scheme is shown on the right in Fig. 5, in which we see that this scheme retains a good level of agreement with the Alouges–Merlet shock curve.

## 6. Conclusion

This paper was devoted to the study of the numerical approximation of one-dimensional nonconservative hyperbolic systems and, more specifically, the problem of constructing numerical schemes which converge to the theoretically specified weak solution. While the path-theory proposed by Dal Maso, LeFloch and Murat's path theory gives a way to rigorously define nonconservative products occurring in these systems, and thereby allows us to define weak solutions to NCHS as well as to derive Rankine–Hugoniot-like jump conditions, the approach does not specify what choices of path are good candidates. Furthermore, the issue of numerically approximating the nonconservative systems is made more complicated since the numerical approximations rarely agree with the theoretical weak solutions near discontinuity waves.

In this paper we investigated numerical schemes constructed using approximate shock curves proposed by Alouges and Merlet. While these curves have interesting mathematical properties in their own right, the fact that they are also a close approximation of the viscous shock profiles of the nonconservative system for a large class of viscosities makes them a well-suited candidate in DLM path-theory. By following the formalism of the well-known Lax–Wendroff theorem, we are motived to consider numerical schemes whose numerical paths $\psi$ locally agree with the Alouges–Merlet shock curves along the shock profiles. We show that a set of shock curves defined using the DLM path-theory will indeed have the same local expansion as the Alouges–Merlet shock curves if and only if they are reversible. We also conjecture that if a numerical scheme has a numerical viscosity which commutes with $A(\mathbf{u})$, and a numerical path which locally agree with the Alouges–Merlet shock curves along a shock profile (or is a Godunov path with reversible shock curves), then the numerical solutions of this scheme will agree with the Alouges–Merlet weak solutions near a discontinuity, to the third order in the strength of the discontinuity. Since the arguments presented here are not rigorous, we leave the formal proof of this conjecture to a future work.

Several numerical results are presented in which we implement the Alouges–Merlet shock curves in DLM path-dependent numerical schemes. In the example presented, the numerical solutions are seen to have a close agreement with the theoretically specified Alouges–Merlet weak solutions. Furthermore, by numerically calculating the shock curve resolved by the Lax–Friedrichs-like scheme, we show that the numerical shock curve agrees closely with the Alouges–Merlet shock curve. We also give examples in which we approximate the numerical paths. These approximations are chosen to have the same local expansion as the Alouges–Merlet shock curves along shock profiles. We show in the examples considered, that the numerical shock curves of these solutions also agree closely with the Alouges–Merlet shock curves. Furthermore, these scheme achieve a good amount of accuracy to the Alouges–Merlet weak solutions while being significantly less computationally expensive.

Although the strategies presented in this paper may help to construct numerical schemes whose numerical solutions will agree closely with the Alouges–Merlet weak solutions, the question of how to construct schemes which converge to the weak solutions associated to an arbitrarily chosen path $\phi$ is still an open problem. More importantly, the question of how to construct schemes which converge to the physically relevant solution is also an open problem. One possible approach is to derive a class of numerical schemes from the weak formulation of NCHSs, using basis functions which are not piecewise constants (as usually done). As the dual space of piecewise constant functions do not entirely contain the space of measure, we expect that the subsequent schemes will in general not converge. Considering now at least continuous functions with compact support (with a support containing the current cell, typically plateau functions), we can expect convergence, as the set of measures is this time contained in the dual space of continuous functions. In general, while this paper offers many insights into the convergence of several numerical schemes for approximating nonconservative systems, the question of proving true convergence is still a topic of future research.

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